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A string field theoretical description of (p, q) minimal superstrings

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Abstract: A string field theory of (p, q) minimal superstrings is constructed with the free-fermion realization of 2-component KP (2cKP) hierarchy, starting from 2-cut ansatz of two-matrix models. Differential operators of 2cKP hierarchy are identified with operators in super Liouville theory, and we obtain algebraic curves for the disk amplitudes of $\eta = -1$ FZZT-branes and the partition functions of neutral/charged $\eta = -1$ ZZ branes, which correctly reproduce those of type 0B (p, q) minimal superstrings in conformal backgrounds. In the course of study, some subtle points are clarified, including a difference of (p, q) even/odd models and quantization of flux, and we show that the Virasoro constraints naturally incorporate quantized fluxes without ambiguity. We also argue within this string field framework that type 0A minimal superstrings can be obtained by orbifolding the type 0B strings with a $\mathbb{Z}_2$ symmetry existing when special backgrounds are taken.

Keywords: Matrix Models, 2D Gravity, Integrable Hierarchies, String Field Theory.

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1. Introduction

Noncritical string theories [1] are useful toy models to investigate various aspects of superstrings. While sharing many of important properties with their critical counterparts, noncritical string theories have fewer degrees of freedom and in many cases they are integrable [2 − 4]. They also have a description based on a string field theory for bosonic cases [5 − 9].

For the last ten years, a great progress has been made in the understanding of noncritical superstring theories based on super Liouville theory [10 − 16], and some exact results on correlation functions have been obtained. The main aim of the present paper is to construct a string field theory of (p, q) minimal superstrings such that it has a nonperturbative definition and correctly reproduces all the known results in super Liouville field theory.

Our basic strategy is to use 2-cut solutions of two-matrix models as a nonperturbative definition of (p, q) minimal superstring theories. This is based on the recent observation that 2-cut solutions [17 − 23] of one-matrix models with symmetric double-well potentials describe two-dimensional pure supergravity, or $(p, q) = (2, 4)$ type 0 superstrings [24, 25]. There the symmetric fluctuations of eigenvalues are interpreted as being in the NS-NS sector while the antisymmetric ones in the R-R sector [26, 27]. Furthermore, by finetuning the potentials, one obtains a series of higher multicritical points, which are identified with $(p, q) = (2, 4k)$ type 0 superstrings [24].[1] This is reminiscent of what happened for bosonic minimal string theory, where one-cut solutions of one-matrix models describe $(p, q) = (2, 2k - 1)$ Kazakov series [28, 29], while those of two-matrix models describe all the (p, q) minimal strings [30, 31]. Thus, it is natural to expect that generic (p, q) minimal superstring theories can be defined as continuum limits of two-matrix models in 2-cut phases. We show that this is indeed the case; we find that 2-cut solutions of two-matrix models generically have the integrable structure of the 2-component KP (2cKP) hierarchy, and that physical operators in super Liouville theory have their counterparts in the 2cKP hierarchy. We will see that this mapping enables us to construct a string field theory of (p, q) minimal superstrings, and the obtained results correctly reproduce those in super Liouville theory.

We here give an intuitive explanation of why 2-cut solutions of matrix models and the 2cKP hierarchy play a particular role in type 0 superstring theory.

Matrix models are originally introduced as the generators of random triangulations of a surface and are found to successfully describe bosonic noncritical strings (or two-dimensional quantum gravity) in their continuum limits. The matrix models of type 0 superstrings do not give triangulations of a surface in a usual sense but behave as the generators of "the square roots of triangulations." This can be seen as follows.

First, from the worldsheet side with Liouville superfield $\Phi(z, \bar{z}, \theta, \bar{\theta}) = \phi(z, \bar{z}) + i\theta\psi(z, \bar{z}) + i\bar{\theta}\bar{\psi}(z, \bar{z}) + i\theta\bar{\theta}F(z, \bar{z})$, the bulk cosmological constants μ is introduced as the coefficients of super Liouville potential (see, e.g., [27]):

$$S_{\rm int}^{(\rm bulk)} \sim \int dz d\bar{z} d\theta d\bar{\theta} \left[-i\mu\, e^{b\Phi(z,\bar{z},\theta,\bar{\theta})}\right] \sim \int dz d\bar{z} \left[-i\mu\, \psi\bar{\psi}\, e^{b\phi} + (\mu^2/4)\, e^{2\,b\,\phi}\right], \qquad (1.1)$$

[1] There are also $(2, 4k + 2)$ theories in a flow generated by R-R operators.

from which one sees that μ is the square root of its bosonic string counterpart, $\mu_{\rm bos}$, the Laplace conjugate to the area of random surface. A similar analysis can be made also for the boundary cosmological constant ζ and its bosonic counterpart $\zeta_{\rm bos}$, and we have the relation

$$\zeta \sim \sqrt{\zeta_{\rm bos}}, \qquad \mu \sim \sqrt{\mu_{\rm bos}}. \tag{1.2}$$

That is, the boundary and bulk cosmological constants of minimal superstrings are related to the square roots of their bosonic counterparts.

On the other hand, the boundary cosmological constant $\zeta_{\rm bos}$ can be regarded as the complex coordinate of two-dimensional target space [7]. In fact, in the bosonic case, the macroscopic operator $\mathcal{O}(\zeta_{\rm bos}) = \displaystyle\int_0^\infty dl\, e^{-l\,\zeta_{\rm bos}}\,\mathcal{O}(l)$ can be expressed (up to the so-called nonuniversal terms) as the summation of microscopic operators $\mathcal{O}_n$

$$\mathcal{O}(\zeta_{\rm bos}) \sim \sum_n c_n\, \mathcal{O}_n\, \zeta_{\rm bos}^{-n/p-1}, \tag{1.3}$$

where $\mathcal{O}_n = \displaystyle\int dzd\bar{z}\, e^{i\alpha_n X(z,\bar{z})+\beta_n\,\phi(z,\bar{z})}$ with the Feigin-Fuchs matter $X(z,\bar{z})$, and $\alpha_n = (b^{-1}-b)-n/\sqrt{pq}$ and $\beta_n = (b^{-1}+b)-n/\sqrt{pq}$ for $c_{\rm matter} = 1-6(q-p)^2/qp$ and $b = \sqrt{p/q}\,(<1)$. The coefficients c_n may be complicated, but in any case $\mathcal{O}(\zeta_{\rm bos})$ receives a nonnegligible contribution when the following relation holds for some $(z,\bar{z})$:

$$\zeta_{\rm bos}^{1/p} \sim e^{-b\,\left(\phi(z,\bar{z})+i\,X(z,\bar{z})\right)}. \tag{1.4}$$

This enables us to interpret the operator $\mathcal{O}(\zeta_{\rm bos})$ as creating a closed loop at the spacetime complex coordinate $\zeta_{\rm bos}$. This also implies that $\zeta_{\rm bos}\in\mathbb{R}_+$ is related to Liouville coordinate ϕ and the limit $\mathrm{Re}\,\zeta_{\rm bos}\to+\infty$ corresponds to the weak coupling region $\phi\to-\infty$ [32].[2]

In super Liouville theory, it is again $\zeta_{\rm bos}\sim\zeta^2$ that allows such analysis based on the Feigin-Fuchs representation. Thus, for a given spacetime Liouville coordinate $\zeta_{\rm bos}\in\mathbb{R}_+$, the boundary cosmological constants ζ and $-\zeta$ are naturally paired to give the bosonic boundary cosmological constant $\zeta_{\rm bos}\sim\zeta^2$, and the weak coupling region of Liouville theory, $\phi\to-\infty$, corresponds to $\mathrm{Re}\,\zeta\to\pm\infty$.

Since the cuts in a spacetime geometry of ζ are also paired to give a single cut for bosonic counterpart $\zeta^2\sim\zeta_{\rm bos}$ (figure 1), such a geometry is realized in a 2-cut hermitian matrix model with a symmetric double-well potential, such that its eigenvalue $x\in\mathbb{R}$ in one cut is always paired with $-x$ in the other cut. It then can be understood that ζ in super Liouville theory is related to the matrix-model eigenvalue x as

$$\zeta = -ix, \tag{1.5}$$

which was first pointed out in [24].

[2]We should stress here that the above is a rough discussion since terms with large n in the summation (1.3) become nonnegligible when $|\zeta_{\rm bos}|$ is small enough which may be out of the perturbative region.

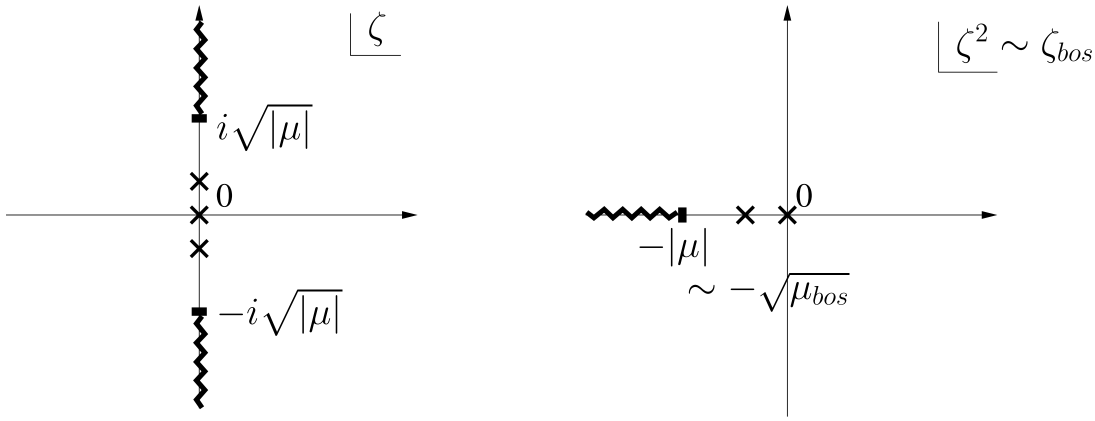

Figure 1: A typical geometry of spacetime with ζ and ζ^2

From the viewpoint of integrable system (or string field theory discussed in this paper), the coordinate $\zeta_{\rm bos}$ in bosonic string theories is represented as the eigenvalue of a differential operator $\boldsymbol{P} = \partial^p + \cdots$ (see, e.g., [9]). Such a system including higher-order differential operators can be systematically analyzed once it is embedded into the KP hierarchy (see, e.g., [33, 34]). For type 0 NSR superstrings, we consider a 2×2 matrix-valued differential operator of the form

$$\boldsymbol{P} = \sigma_3\, \partial^{\hat{p}} + \cdots \tag{1.6}$$

which naturally has a pair of eigenfunctions $\Psi^{(1)}$ and $\Psi^{(2)}$ with eigenvalues ζ and $-\zeta$, respectively:

$$\boldsymbol{P}\,\Psi^{(1)} = \zeta\,\Psi^{(1)}, \qquad \boldsymbol{P}\,\Psi^{(2)} = -\zeta\,\Psi^{(2)}. \tag{1.7}$$

This is reminiscent of the Dirac theory of fermions, where the "square root" of the Klein-Gordon equation is realized by using a matrix-valued differential operator. Such a system turns out to be in a class of 2cKP hierarchy as we review in the next section. There we will find that the operators of the form $\boldsymbol{O}_n^{[0]} = \partial^n + \cdots$ correspond to the fluctuations of an NS-NS scalar, and those of the form $\boldsymbol{O}_n^{[1]} = \sigma_3\,\partial^n + \cdots$ to the fluctuations of an R-R scalar. We will also see that such system can be fully described by 2-component fermions,

$$\big(c_0^{(1)}(\zeta), \bar{c}_0^{(1)}(\zeta)\big), \quad \big(c_0^{(2)}(\zeta), \bar{c}_0^{(2)}(\zeta)\big), \tag{1.8}$$

the former of which corresponds to the creation/annihilation of an $\eta = -1$ FZZT brane of charge $+1/2$ and the latter to that of charge $-1/2$, both placed at the same spacetime point ζ^2 but at different "superspace points" ζ and $-\zeta$, respectively.[3]

[3]Here η represents the relation of the left and right supercharges on a boundary, $Q_{\rm L} = i\eta\, Q_{\rm R}$. The operators that can be naturally derived from matrix models are found to choose $\eta = -1$, as was pointed out in [24]. It would be interesting to investigate whether our string field theory can describe the case $\eta = +1$. There is an interesting proposal from a loop gas approach [35].

This paper is organized as follows. In section 2, we analyze two-cut two-matrix models closely following the analysis made in one-cut two-matrix models [30, 31, 36], and show that operators in super Liouville field theory can totally be written in terms of 2cKP hierarchy. In section 3, we solve the Douglas equation in such systems and derive the $W_{1+\infty}$ constraints. We also derive string equations, and show that background R-R fluxes can be incorporated into our formalism without ambiguity (not as integration constants in string equations). This reflects the fact that the Virasoro constraints (included in the $W_{1+\infty}$ constraints) are in the "once-integrated form" of string equations [37]. In section 4, we introduce type 0B (p, q) minimal string field theory, and clarify the meaning of FZZT and ZZ branes in our context. In section 5, we calculate the disk amplitudes of $\eta = -1$ FZZT branes and derive the corresponding algebraic curves. We find there that the even and odd minimal superstrings, though apparently different, can have a common description. We also calculate the partition functions of neutral/charged $\eta = -1$ ZZ branes. Section 6 is devoted to conclusion and discussion, and we there make a comment on how type 0A superstrings (which is described by complex matrix models [38−41]) are obtained within our framework.

2. 2-cut two-matrix models and 2cKP hierarchy

In this section, we investigate 2-cut two-matrix models and show that 2-cut ansatz gives 2-component KP (2cKP) hierarchy [33, 42, 43] as their integrable structure. We also establish the identification of the differential operators appearing in 2cKP hierarchy with the operators in (p, q) minimal superstring theories.

2.1 Two-matrix models with 2-cut ansatz

The partition function of a two-matrix model is defined by the following integral of two $N \times N$ hermitian matrices X and Y:

$$Z_{\mathrm{lat}} \equiv \int dX dY e^{-N \operatorname{tr} w(X,Y)}, \quad w(X, Y) \equiv V_1(X) + V_2(Y) - cXY, \tag{2.1}$$

and can be written in terms of the eigenvalues of X and Y ($\{x_i\}$ and $\{y_i\}$ $(i = 1, \cdots, N)$, respectively) as

$$Z_{\mathrm{lat}} = \int \prod_{i=1}^{N} dx_i\, dy_i\, \Delta(x)\, \Delta(y)\, e^{-N \sum_i w(x_i, y_i)}. \tag{2.2}$$

Here $\Delta(x)$ and $\Delta(y)$ are the van der Monde determinants $\big(\text{e.g.}\, \Delta(x) = \prod_{i<j}(x_i - x_j)\big)$. Since 2-cut critical points of one-matrix models were found at the origin of $\mathbb{Z}_2$ symmetric potential of eigenvalue λ, $V(-\lambda) = V(\lambda)$ [17−19], it is natural to expect that 2-cut critical points of two-matrix models are obtained from the following $\mathbb{Z}_2$ symmetric potential[4]

$$w(-x, -y) = w(x, y), \tag{2.3}$$

[4]In a similar way, we can introduce multi-cut, multi-critical multi-matrix models with the strategy of [44]. For example, 4-cut two-matrix models can be derived by requiring the following $\mathbb{Z}_4$ symmetry: $w(\omega x, \omega^3 y) = w(x, y)$, $(\omega = e^{2\pi i/4})$, under which the interaction term cxy is invariant.

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or equivalently $V_1(-x) = V_1(x)$ and $V_2(-y) = V_2(y)$. In this paper, we consider more general cases where systems may also be under $\mathbb{Z}_2$-odd perturbations, and study the two-matrix potential with the following $\mathbb{Z}_2$ property:

$$w(-x,-y) = w^*(x,y), \tag{2.4}$$

where * means complex conjugation. This complex conjugation implies that coefficients of odd terms in $V_1(x)$ and $V_2(y)$ are pure imaginary. This follows from the one-matrix-model analysis [21], where critical points are found on flows of $\mathbb{Z}_2$-odd perturbations only when the coefficients of odd terms are pure imaginary. It is an analogue of Lee-Yang edge singularity [29], and matrix models with this setup are known to be defined nonperturbatively with potentials bounded from below. We assume that it is also the case in two-matrix models.

This model can be solved by investigating its orthogonal polynomial system

$$\alpha_n(x) = \frac{1}{\sqrt{h_n}}\big(x^n+\cdots\big), \quad \beta_n(x) = \frac{1}{\sqrt{h_n}}\big(y^n+\cdots\big) \qquad (n=0,1,2,\cdots), \tag{2.5}$$

which we now require to satisfy the orthonormality conditions:

$$\delta_{m,n} = \big\langle \alpha_m \big| \beta_n \big\rangle \equiv \int dx\,dy\, e^{-Nw(x,y)}\,\alpha_m(x)\,\beta_n(y), \tag{2.6}$$

and the partition function is given as $Z_{\mathrm{lat}} = N!\,\prod_{n=0}^{N-1} h_n$. Reflecting the above $\mathbb{Z}_2$ symmetry (2.4), the orthonormal polynomials $\alpha_n(x)$ and $\beta_n(y)$ are equipped with the following $\mathbb{Z}_2$ structure:

$$\alpha_n(-x) = (-1)^n\,\alpha_n^*(x), \quad \beta_n(-y) = (-1)^n\,\beta_n^*(y) \qquad (n=0,1,2,\cdots). \tag{2.7}$$

Then the 2-cut critical points are realized at the origin of eigenvalues, $(x_c,y_c)=(0,0)$.[5]

This polynomial system is characterized by the canonical pairs of operators $\boldsymbol{Q_1}$, $\boldsymbol{P_1}$, $\boldsymbol{Q_2}$ and $\boldsymbol{P_2}$ defined by

$$x\,\alpha_n(x) = \sum_m \alpha_m(x)\,\big(\boldsymbol{Q_1}\big)_{mn}, \qquad \frac{d}{dx}\alpha_n(x) = \sum_m \alpha_m(x)\,\big(\boldsymbol{P_1}\big)_{mn}, \tag{2.8}$$

$$y\,\beta_n(y) = \sum_m \beta_m(y)\,\big(\boldsymbol{Q_2}\big)_{mn}, \qquad \frac{d}{dy}\beta_n(y) = \sum_m \beta_m(y)\,\big(\boldsymbol{P_2}\big)_{mn}, \tag{2.9}$$

which satisfy the relations

$$\big[\boldsymbol{P_1},\,\boldsymbol{Q_1}\big] = \boldsymbol{1}, \qquad \big[\boldsymbol{P_2},\,\boldsymbol{Q_2}\big] = \boldsymbol{1}. \tag{2.10}$$

They are equivalent to

$$\big[\boldsymbol{Q_1}^{\mathrm{T}}, -c\boldsymbol{Q_2}\big] = N^{-1}\,\boldsymbol{1}, \tag{2.11}$$

[5] More generally, one can assume $(x_c,y_c)=(ic_1,ic_2)$, but this always can be shifted to the origin by a proper deformation of potential, which corresponds to redundant operators in the R-R sector.

since $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$ satisfy

$$\boldsymbol{P_1} = N\big(-c\,\boldsymbol{Q_2}^{\mathrm{T}} + V_1'(\boldsymbol{Q_1})\big), \quad \boldsymbol{P_2} = N\big(-c\,\boldsymbol{Q_1}^{\mathrm{T}} + V_2'(\boldsymbol{Q_2})\big). \tag{2.12}$$

Since the scaling behaviours of these polynomials are different for even and odd index n, we rearrange the orthonormal basis such that n runs as $n = 0, 2, 4, \cdots, 1, 3, 5, \cdots$ and multiply the basis $(\alpha_{2k}(x),\, \alpha_{2k+1}(x))$ and $(\beta_{2k}(y),\, \beta_{2k+1}(y))$ by a common factor $(-1)^k$ [44]. Then the operators $\boldsymbol{Q}_1^{\mathrm{T}}$ and $\boldsymbol{Q}_2$ become $2{\times}2$ block matrices, each of four elements is a difference operator, and we can render these difference operators such as to behave smoothly for a small change of n by tuning the potentials properly.

As is discussed in appendix A in detail we can take a continuum limit in such a way that they have the following scaling behaviors with respect to the lattice spacing a of random surfaces:

$$N^{-1} = g\, a^{(\hat{p}+\hat{q})/2}, \qquad \boldsymbol{Q_1} = ic_1\mathbf{1}_2 + i\, a^{\hat{p}/2}\,\boldsymbol{P}, \qquad \boldsymbol{Q_2}^{\mathrm{T}} = ic_2\mathbf{1}_2 + i\, a^{\hat{q}/2}\,\boldsymbol{Q}. \tag{2.13}$$

Here $\boldsymbol{P}$ and $\boldsymbol{Q}$ are 2×2 matrix-valued differential operators of order $\hat{p}$ and $\hat{q}$, respectively, with respect to the scaling variable $\xi \equiv -a^{-(\hat{p}+\hat{q}-1)/2}\dfrac{N-n}{N}$ (<0):

$$\boldsymbol{P} = \sum_{i=0}^{\hat{p}} U_i^P(\xi)\,\partial^{\hat{p}-i}, \quad \boldsymbol{Q} = \sum_{j=0}^{\hat{q}} U_j^Q(\xi)\,\partial^{\hat{q}-j} \qquad \Big(\partial \equiv g\,\frac{\partial}{\partial\xi} = -a^{-1/2}\,\frac{\partial}{\partial n}\Big). \tag{2.14}$$

The analysis made in appendix A allows us to set the following reality condition on the coefficients

$$U_k^P,\ U_k^Q \in \mathbb{R}\,\mathbf{1}_2 + \mathbb{R}\,\sigma_3 + \mathbb{R}\,\sigma_2 + i\,\mathbb{R}\,\sigma_1, \tag{2.15}$$

especially $U_0^P,\, U_0^Q \in \mathbb{R}\,\mathbf{1}_2 + \mathbb{R}\,\sigma_2$. Then eq. (2.11) is rewritten into the form of the Douglas equation [30]

$$\big[\boldsymbol{P},\, \boldsymbol{Q}\big] = g\,\mathbf{1}. \tag{2.16}$$

The Douglas equation (2.16) is invariant under the transformation

$$\boldsymbol{P} \to c\,V(\xi)\cdot\boldsymbol{P}\cdot V(\xi)^{-1}, \qquad \boldsymbol{Q} \to c^{-1}\,V(\xi)\cdot\boldsymbol{Q}\cdot V(\xi)^{-1} \tag{2.17}$$

with a nonvanishing constant c and a regular function of ξ, $V(\xi) = e^{\,\mathbf{1}_2 f_1(\xi) + \sigma_3 f_2(\xi)}\, e^{i(\pi/4)\sigma_1}$. Using this, we can always assume that the pair $(\boldsymbol{P}, \boldsymbol{Q})$ is in the canonical form which satisfies:

$$U_0^P = \sigma_3^{\epsilon} e^{\theta_p\sigma_3}, \quad U_0^Q = c_q\,(\sigma_3\cos\theta_q + \mathbf{1}_2\sin\theta_q), \quad U_1^P = \begin{pmatrix} 0 & U_+^P \\ U_-^P & 0 \end{pmatrix}. \tag{2.18}$$

The case of $\theta_p = 0$ and $\epsilon = 1$ is of our concern as was discussed in Introduction, and we assume this in the rest of this article:[6]

$$\boldsymbol{P} = \sigma_3\,\partial^{\hat{p}} + \cdots, \quad \boldsymbol{Q} = \big(c_q^{[0]}\sigma_3 + c_q^{[1]}\mathbf{1}_2\big)\partial^{\hat{q}} + \cdots. \tag{2.19}$$

[6] This assumption can actually be set without loss of generality. See footnote 18 in subsection 3.2.

It turns out that the special cases where $c_q^{[1]}=0$ correspond to critical points obtained from $\mathbb{Z}_2$ symmetric potentials (i.e. universality classes defined by NS-NS operators), and that the cases where $c_q^{[0]}=0$ correspond to critical points to be found in odd perturbations (i.e. in a flow generated by R-R operators).

2.2 Deformation of potentials and 2cKP hierarchy

Under deformations of the potential $w(x,y)$ in two-matrix models

$$N\,w(x,y)\to N\,w(x,y)+N\,\delta w(x,y), \tag{2.20}$$

the matrix-valued differential operators $\boldsymbol{P}$ and $\boldsymbol{Q}$ will change as

$$\delta\boldsymbol{P}=\frac{1}{g}\big[\boldsymbol{H},\boldsymbol{P}\big],\qquad \delta\boldsymbol{Q}=\frac{1}{g}\big[\boldsymbol{H},\boldsymbol{Q}\big], \tag{2.21}$$

with retaining the Douglas equation (2.16). An analysis similar to the 1-cut case [33, 45] shows that such $\boldsymbol{H}$ can be expanded as[7]

$$\boldsymbol{H}=\sum_{n=0}^{\infty}\sum_{\mu=0,1}\delta x_n^{[\mu]}\,\big(\boldsymbol{\sigma}^{\mu}\boldsymbol{L}^{n}\big)_{+}. \tag{2.22}$$

Here $\boldsymbol{\sigma}$ and $\boldsymbol{L}$ are matrix-valued pseudo-differential operators of the form

$$\boldsymbol{\sigma}=\sigma_3+\sum_{n=1}^{\infty}Z_n(\xi)\,\partial^{-n},\quad \boldsymbol{L}=\mathbf{1}_2\,\partial+\sum_{n=1}^{\infty}U_{n+1}(\xi)\,\partial^{-n} \tag{2.23}$$

which satisfy the relations[8]

$$\big[\boldsymbol{\sigma},\boldsymbol{P}\big]=0,\quad \boldsymbol{\sigma}^2=\mathbf{1}_2,\quad \boldsymbol{\sigma}\boldsymbol{L}^{\hat p}=\boldsymbol{P}, \tag{2.24}$$

and the positive and negative parts of a matrix-valued pseudo-differential operator $\boldsymbol{A}=\sum_{n\in\mathbb{Z}}a_n\partial^n$ are defined as

$$\boldsymbol{A}_{+}\equiv\sum_{n\geq 0}a_n\partial^n,\qquad \boldsymbol{A}_{-}\equiv\sum_{n<0}a_n\partial^n. \tag{2.25}$$

One can easily see that the operators of $(\boldsymbol{L}^n)_+$ $(\mu=0)$ are given by even deformations of the potential $w(x,y)$ while $(\boldsymbol{\sigma}\boldsymbol{L}^n)_+$ $(\mu=1)$ by odd deformations.

By assuming that the pair of differential operators $(\boldsymbol{P},\boldsymbol{Q})$ are under perturbations with finite $x_n^{[\mu]}$'s, they become functions of the variables $x=(x_n^{[\mu]})$ $(n=1,2,\cdots;\,\mu=0,1)$, and the deformation equations are rewritten as

$$g\,\frac{\partial\boldsymbol{P}}{\partial x_n^{[\mu]}}=\big[(\boldsymbol{\sigma}^{\mu}\boldsymbol{L}^{n})_{+},\boldsymbol{P}\big],\qquad g\,\frac{\partial\boldsymbol{Q}}{\partial x_n^{[\mu]}}=\big[(\boldsymbol{\sigma}^{\mu}\boldsymbol{L}^{n})_{+},\boldsymbol{Q}\big]. \tag{2.26}$$

[7]By requiring that $\boldsymbol{P}+\delta\boldsymbol{P}$ and $\boldsymbol{Q}+\delta\boldsymbol{Q}$ also retain the $\mathbb{Z}_2$ structure (2.4), all the parameter $\delta x_n^{[\mu]}$ must be real, and thus $\boldsymbol{H}$ also satisfies the condition (2.15).

[8]In this paper, $\boldsymbol{\sigma}^{\mu}$ means $(\boldsymbol{\sigma})^{\mu}$, and the index of the Pauli matrix is denoted by subscript: $\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$ and $\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. For example, $\sigma_3^2=(\sigma_3)^2=1$ etc.

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One can easily find that the operators $\boldsymbol{\sigma}$ and $\boldsymbol{L}$ are uniquely determined for a given differential operator $\boldsymbol{P}$, and thus the deformation equations of $\boldsymbol{\sigma}$ and $\boldsymbol{L}$ are given by

$$g\,\frac{\partial \boldsymbol{\sigma}}{\partial x_n^{[\mu]}} = \big[(\boldsymbol{\sigma}^\mu \boldsymbol{L}^n)_+, \boldsymbol{\sigma}\big], \qquad g\,\frac{\partial \boldsymbol{L}}{\partial x_n^{[\mu]}} = \big[(\boldsymbol{\sigma}^\mu \boldsymbol{L}^n)_+,\, \boldsymbol{L}\big]. \tag{2.27}$$

These are known as the Lax equations of 2cKP hierarchy. Thus we conclude that the underlying integrable structure of 2-cut two-matrix models is the 2cKP hierarchy. Note that $\partial\,\big(\equiv \partial/\partial\xi\big) = g\,\partial/\partial x_1^{[0]}$ since $\boldsymbol{L}_+ = \mathbf{1}_2\,\partial$. Thus, all of the above operators depend on the indeterminate ξ as $f(\xi, x) = f(x_1^{[0]} + g\xi, x_2^{[0]}, \cdots;\, x_1^{[1]}, x_2^{[1]}, \cdots)$. Therefore, in the rest of the present paper, we absorb ξ into $x_1^{[0]}$ and set $\xi = 0$.

The 2cKP hierarchy equations (2.27) are the conditions that ensure the existence of a matrix-valued Baker-Akhiezer function $\Psi(x;\lambda) = \big(\Psi^{(ij)}(x;\lambda)\big)$ $(i,j = 1,2)$ solving the following linear problem with a spectral parameter λ which is invariant under the flows of $x_n^{[\mu]}$'s:

$$\boldsymbol{\sigma}\,\Psi(x;\lambda) = \Psi(x;\lambda)\,\sigma_3, \qquad \boldsymbol{L}\,\Psi(x;\lambda) = \Psi(x;\lambda)\,\Lambda, \qquad \left(\Lambda \equiv \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) \tag{2.28}$$

and

$$g\,\frac{\partial \Psi(x;\lambda)}{\partial x_n^{[\mu]}} = (\boldsymbol{\sigma}^\mu \boldsymbol{L}^n)_+ \Psi(x;\lambda). \tag{2.29}$$

The linear problem and the Douglas equation can be best solved by introducing a 2×2 matrix-valued Sato-Wilson operator $\boldsymbol{W}(x;\partial) = \mathbf{1}_2 + \sum_{n\geq 1} w_n(x)\partial^{-n}$, which satisfies

$$\boldsymbol{\sigma} = \boldsymbol{W}\sigma_3\boldsymbol{W}^{-1}, \quad \boldsymbol{L} = \boldsymbol{W}\partial\boldsymbol{W}^{-1}. \tag{2.30}$$

The 2cKP equations (2.27) are equivalent to the so-called Sato equations

$$g\,\frac{\partial \boldsymbol{W}}{\partial x_n^{[\mu]}} = (\boldsymbol{\sigma}^\mu \boldsymbol{L}^n)_+ \boldsymbol{W} - \boldsymbol{W}\sigma_3^\mu \partial^n \;\big(= -(\boldsymbol{\sigma}^\mu \boldsymbol{L}^n)_- \boldsymbol{W}\big). \tag{2.31}$$

Then the Baker-Akhiezer function is solved as

$$\begin{aligned}\Psi(x;\lambda) &= \boldsymbol{W}(x;\partial)\cdot \exp\Big(g^{-1}\sum_{n\geq 1}\sum_{\mu=0,1} \lambda^n x_n^{[\mu]}\sigma_3^\mu\Big) \\ &= \Phi(x;\lambda)\cdot \exp\Big(g^{-1}\sum_{n\geq 1}\sum_{\mu=0,1} \lambda^n x_n^{[\mu]}\sigma_3^\mu\Big)\end{aligned} \tag{2.32}$$

with $\Phi(x;\lambda) \equiv \boldsymbol{W}(x;\partial)\big|_{\partial\to\lambda}$. Note that the operator $\boldsymbol{W}$ is uniquely determined up to the right-multiplication of a diagonal pseudo-differential operator with constant coefficients: $\boldsymbol{W} \to \boldsymbol{W}\cdot e^{\sum_{n<0} c_n \partial^n}$ $(c_n \in \mathbb{R}\,\mathbf{1}_2 + \mathbb{R}\,\sigma_3)$.

2.3 Identification with (p,q) minimal superstring theory

We now identify the operators of 2cKP hierarchy (or the matrix models) with those of super Liouville theory. There are two types of minimal superstring theories (see, e.g., [24]):

(p,q) **even minimal superstrings**: $(p,q)=(2\hat{p},2\hat{q})$ with $\hat{p}+\hat{q}\in 2\mathbb{Z}+1$,

(p,q) **odd minimal superstrings**: $(p,q)=(\hat{p},\hat{q})$ with $\hat{p},\,\hat{q}\in 2\mathbb{Z}+1$ $(\hat{p}+\hat{q}\in 2\mathbb{Z})$.

Each has a sequence of operators $\mathcal{O}_n^{[r-s]}$ $(n\equiv\hat{q}r-\hat{p}s)$, where $r-s\in 2\mathbb{Z}$ corresponds to NS-NS operators and $r-s\in 2\mathbb{Z}+1$ to R-R operators.[9] They are expressed as the gravitational dressing of the corresponding operators $\mathcal{O}_n^{(\mathrm{M})[r-s]}$ of (p,q) minimal superconformal matters which have the conformal dimensions

$$\Delta^{(\mathrm{M})[r-s]}_{n=(\hat{q}r-\hat{q}s)}=\frac{(\hat{q}r-\hat{p}s)^2-(\hat{q}-\hat{p})^2}{8\hat{q}\hat{p}}+\frac{1-(-1)^{r-s}}{32}. \tag{2.33}$$

The string susceptibility γ_{str} is measured with the operator $\mathcal{O}_{\mathrm{min}}$ of minimal scaling dimension which is given by either of $\mathcal{O}_1^{[0]}$ (NS-NS) or $\mathcal{O}_1^{[1]}$ (R-R) for even minimal superstrings and by $\mathcal{O}_{\mathrm{min}}=\mathcal{O}_1^{[1]}$ (R-R) for odd minimal superstrings, and is found to be

$$\gamma_{\mathrm{str}}=-\frac{1}{\hat{p}+\hat{q}-1}=\begin{cases}-\dfrac{2}{p+q-2} & \text{(even)}\\ -\dfrac{1}{p+q-1} & \text{(odd)}\end{cases}. \tag{2.34}$$

The gravitational scaling dimensions $\Delta_n^{[\mu]}$ of the operators $\mathcal{O}_n^{[\mu]}$ are then defined by

$$\frac{\left\langle\mathcal{O}_{n_1}^{[\mu_1]}\cdots\mathcal{O}_{n_N}^{[\mu_N]}\,e^{-t\,\mathcal{O}_{\mathrm{min}}}\right\rangle}{\left\langle\mathcal{O}_{\mathrm{min}}\cdots\mathcal{O}_{\mathrm{min}}\,e^{-t\,\mathcal{O}_{\mathrm{min}}}\right\rangle}\equiv|t|^{\sum_{k=1}^{N}\Delta_{n_k}^{[\mu_k]}}, \tag{2.35}$$

and given by

$$\Delta_n^{[0]}=\Delta_n^{[1]}=\frac{\hat{p}+\hat{q}-n}{\hat{p}+\hat{q}-1}. \tag{2.36}$$

To identify the operators, we first note that the operators $(\boldsymbol{L}^n)_+$ and $(\boldsymbol{\sigma}\boldsymbol{L}^n)_+$ are differential operators of the same order. They will give the same gravitational scaling dimensions and thus should be considered in pairs. Furthermore, according to the analysis of one-matrix model with a symmetric double-well potential, the excitations symmetric under the reflection belong to the NS-NS sector, and those antisymmetric to the R-R sector [26, 27, 24]. Following the derivation given in subsection 2.1, one can easily see that such reflection is realized in our 2cKP by the transformation $\mathrm{C}:(\boldsymbol{\sigma},\boldsymbol{L})\to(-\boldsymbol{\sigma},\boldsymbol{L})$.[10] This implies that in the pair $\big((\boldsymbol{L}^n)_+,\,(\boldsymbol{\sigma}\boldsymbol{L}^n)_+\big)$, the former $(\mu=0)$ should belong to the NS-NS sector and the latter $(\mu=1)$ to the R-R sector. This consideration naturally leads us to the following ansatz on the operator identification:

$$\mathcal{O}_n^{[r-s]}\ \leftrightarrow\ \big(\boldsymbol{\sigma}^{r-s}\boldsymbol{L}^n\big)_+=\sigma_3^{r-s}\,\partial^n+\cdots. \tag{2.37}$$

See figure 2 (a) and (b1) for the Kac table of the $(p,q)=(2\hat{p},2\hat{q})=(4,6)$ and $(p,q)=(\hat{p},\hat{q})=(3,5)$ minimal superconformal matters.

[9] $[r-s]$ implies $r-s$ modulo 2.

[10] This charge conjugation will be studied more in subsection 3.3.

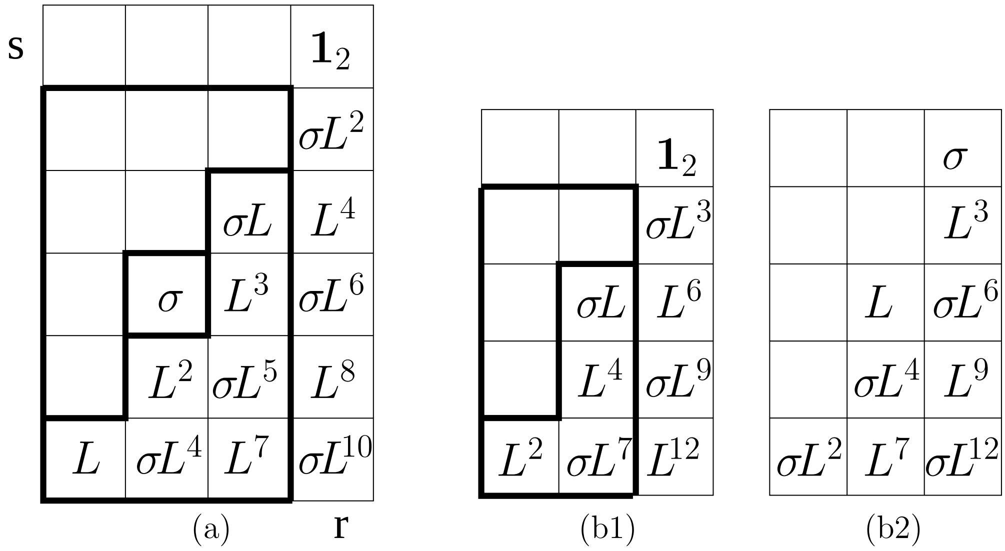

Figure 2: Kac table for $(p,q) = (4,6)$ and $(p,q) = (3,5)$ minimal conformal matters. Letters in boxes denote the corresponding differential operators $(\boldsymbol{\sigma}^\mu \boldsymbol{L}^n)_+ \leftrightarrow \mathcal{O}^{[r-s]}_{\hat{q}r-\hat{p}s}$. The usual conformal block of minimal conformal matters is shown by a rectangular with bold lines. The table (b2) expresses those differential operators of $(3,5)$ model that do not have their counterparts in Liouville theory. They are all odd under the $\mathbb{Z}_2$ transformation, $\tilde{\mathrm{C}} : (\boldsymbol{\sigma}, \boldsymbol{L}) \to (-\boldsymbol{\sigma}, -\boldsymbol{L})$.

We are now in a position to discuss that some operators do not appear in the spectrum. First, the operators $\boldsymbol{P}^n = (\boldsymbol{P}^n)_+$ do not have their counterparts $\mathcal{O}^{[n]}_{n\hat{p}}$ in the Kac table of minimal conformal matters. However, as can be seen in (2.26) or in (2.27), the evolution along these directions is trivial, so that correlation functions including such operators always vanish.

Second, while the 2cKP hierarchy allows the operators $(\boldsymbol{\sigma}^{n+1}\boldsymbol{L}^n)_+$, they do not have their counterparts in the Kac table of (p,q) odd minimal conformal matters (figure 2 (b2)). In this case one can reduce the system by orbifolding it with another $\mathbb{Z}_2$ symmetry, $\tilde{\mathrm{C}}$: $(\boldsymbol{\sigma}, \boldsymbol{L}) \to (-\boldsymbol{\sigma}, -\boldsymbol{L})$, after which we have a complete one-to-one correspondence between the operators of 2cKP hierarchy and those of super Liouville theory. We study this orbifolding in detail in subsection 3.3.

We here comment that the spectrum of type 0A (p,q) minimal superstrings can be obtained by orbifolding those of type 0B minimal superstrings with the $\mathbb{Z}_2$ transformation C. This is because there are no R-R fields in 0A minimal theories other than the degrees of freedom of R-R fluxes [24]. We also comment that there exist a series of critical points characterized by $\boldsymbol{Q} = c_q^{[1]}\partial^{\hat{q}} + \cdots$, which can be obtained by perturbing critical systems with R-R operators. The $\tilde{\mathrm{C}}$ invariance holds when $\hat{p}+\hat{q} \in 2\mathbb{Z}+1$ at the new fixed points.

2.4 Grassmannian of 2cKP hierarchy and free fermions

In this subsection, we summarize basic properties of 2cKP hierarchy and its free-fermion realization.

We introduce another set of evolution parameters $x_n^{(i)}$ $(i=1,2)$ which are related to $x_n^{[\mu]}$ $(\mu=0,1)$ as

$$x_n^{(i)} = x_n^{[0]} + (-1)^{i-1} x_n^{[1]} \quad (i=1,2), \tag{2.38}$$

or

$$x_n^{[\mu]} = \frac{1}{2}\left(x_n^{(1)} + (-1)^{\mu} x_n^{(2)}\right) \quad (\mu=0,1). \tag{2.39}$$

Recall that $x_n^{[0]}$ (or $x_n^{[1]}$) correspond to the background sources for NS-NS (or R-R) operators $\mathcal{O}_n^{[0]}$ (or $\mathcal{O}_n^{[1]}$), which, in the language of matrix models with a symmetric double-well potential, describe symmetric (or antisymmetric) excitations of eigenvalues. Thus, the operators $\mathcal{O}_n^{(i)} \equiv (1/2)\big(\mathcal{O}_n^{[0]} + (-1)^{i-1}\mathcal{O}_n^{[1]}\big)$ with $i=1$ (or $i=2$) describes the excitations of the eigenvalues in the left (or right) well.

The Sato equations (2.31) are then expressed as

$$g\,\frac{\partial \boldsymbol{W}}{\partial x_n^{(i)}} = (\boldsymbol{e}^{(i)}\boldsymbol{L}^n)_+\boldsymbol{W} - \boldsymbol{W}E^{(i)}\partial^n \qquad (i=1,2), \tag{2.40}$$

where

$$\boldsymbol{e}^{(i)} = \boldsymbol{W}E^{(i)}\boldsymbol{W}^{-1}, \qquad E^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.41}$$

For a given solution $\boldsymbol{W}(x;\partial)$ to the Sato equations, we define a series of 2×2 matrix-valued functions of λ, $\Phi_K(x;\lambda) = \big(\Phi_K^{(ij)}(x;\lambda)\big)$ $(K=0,1,2,\cdots)$, as

$$\Phi_K(x;\lambda) \equiv e^{-(1/g)x_1^{[0]}\lambda}\cdot\partial^K\cdot\boldsymbol{W}(x;\partial)\cdot e^{(1/g)x_1^{[0]}\lambda} = \left[\partial^K\cdot\boldsymbol{W}(x;\partial)\right]\Big|_{\partial\to\lambda}. \tag{2.42}$$

Note that $\Phi_{K=0}(x;\lambda) = \Phi(x;\lambda)$ (see eq. (2.32)).

In order to interpret the Sato equations as describing the motion in an infinite dimensional Grassmannian $\mathcal{M}$ [33], we first introduce an infinite dimensional vector space $\mathcal{H}$ consisting of 2-component row vectors $\boldsymbol{v}(\lambda) = (v^{(1)}(\lambda), v^{(2)}(\lambda))$ whose components are functions of λ. We also introduce the subspace $\mathcal{V}_0$ which is spanned by the vectors $\boldsymbol{v}_K^{(1)}(\lambda) = (\lambda^K, 0)$ $(K\geq 0)$ and $\boldsymbol{v}_K^{(2)}(\lambda) = (0,\lambda^K)$ $(K\geq 0)$. The infinite dimensional Grassmannian $\mathcal{M}$ is then defined as the set of those subspaces of $\mathcal{H}$ that have the same "size" with $\mathcal{V}_0$:[11]

$$\mathcal{M} \equiv \big\{\mathcal{V}\subset\mathcal{H}\,|\,\mathcal{V}\sim\mathcal{V}_0\big\}. \tag{2.43}$$

For a given Sato-Wilson operator $\boldsymbol{W}(x,\partial)$ and the corresponding sequence of matrices $\Phi_K(x;\lambda)$ given in (2.42), we introduce a point $\mathcal{V}(x)\in\mathcal{M}$ as such that is spanned by a series of row vectors $\boldsymbol{\Phi}_K^{(i)}(x;\lambda) \equiv \big(\Phi_K^{(i1)}(x;\lambda),\,\Phi_K^{(i2)}(x;\lambda)\big)$ $(K\geq 0;\, i=1,2)$. We denote this by using the 2×2 matrices $\Phi_K(x;\lambda) = \big(\boldsymbol{\Phi}_K^{(1)}(x;\lambda),\boldsymbol{\Phi}_K^{(2)}(x;\lambda)\big)^{\mathrm{T}}$ $(K\geq 0)$ as[12]

$$\mathcal{V}(x) \equiv \big\langle\Phi_K(x;\lambda)\big\rangle_{K\geq 0}. \tag{2.44}$$

[11] See, e.g., [33, 45, 42, 43] for a more rigorous definition.

[12] $\mathcal{V}_0$ is then expressed as $\left\langle\begin{pmatrix}\lambda^K & 0\\ 0 & \lambda^K\end{pmatrix}\right\rangle_{K\geq 0}$ and corresponds to the Sato-Wilson operator $\boldsymbol{W}=\mathbf{1}_2$.

The x-evolutions of the linear subspace $\mathcal{V}(x)$ can then be read off by looking at those of the matrices $\Phi_K(x;\lambda)$ $(K \geq 0)$:

$$g\,\frac{\partial \Phi_K(x;\lambda)}{\partial x_n^{(i)}} = -\,\Phi_K(x;\lambda)\,\lambda^n E^{(i)} + \Big[\partial^K \cdot (\boldsymbol{e}^{(i)}\boldsymbol{L}^n)_+ \cdot \boldsymbol{W}\Big]\Big|_{\partial\to\lambda}. \tag{2.45}$$

The second term on the right-hand side simply generates a linear transformation among the vectors $\big\{\boldsymbol{\Phi}_l^{(i)}(x;\lambda)\big\}_{l\leq k,\, i=1,2}$, and thus can be ignored in considering the motion of $\mathcal{V}(x)$. Integrating this equation, we obtain

$$\mathcal{V}(x) = \big\langle \Phi_K(0;\lambda)\, e^{-(1/g)\sum_{n,i} x_n^{(i)}\cdot\lambda^n E^{(i)}} \big\rangle_{K\geq 0} \equiv \mathcal{V}(0)\cdot e^{-(1/g)\sum_{n,\mu} x_n^{(i)}\cdot\lambda^n E^{(i)}}. \tag{2.46}$$

Here $\mathcal{V}(0)$ is the subspace in $\mathcal{H}$ corresponding to the initial value of $\boldsymbol{W}$ at $x=0$.

The free-fermion realization is obtained by making a map from this Grassmannian to the Fock space of a free-fermion system. Because the linear space $\mathcal{V}(x)$ is spanned by 2-component vector-valued functions $\boldsymbol{\Phi}_K^{(i)}(x;\lambda)$ $(K\geq 0;\ i=1,2)$, we need two pairs of free chiral fermions on the complex λ plane, $(\psi^{(i)}(\lambda), \bar\psi^{(i)}(\lambda))$, $(i=1,2)$ [42]. Then a point $\mathcal{V}(x)\in\mathcal{M}$ is assigned with the following state $\big|\Phi(x)\big\rangle$ in the fermion Fock space:

$$\big|\Phi(x)\big\rangle \equiv \prod_{K\geq 0}\prod_{i=1,2}\left[\sum_{j=1,2}\oint\frac{d\lambda}{2\pi i}\,\Phi_K^{(ij)}(x;\lambda)\,\bar\psi^{(j)}(\lambda)\right]\big|\Omega\big\rangle. \tag{2.47}$$

The sum over j $(=1,2)$ reflects the fact that $\boldsymbol{\Phi}_K^{(i)} = (\Phi_K^{(i1)}, \Phi_K^{(i2)})$ is a two-component vector.

We introduce free chiral bosons

$$\begin{aligned}\phi^{(i)}(\lambda) &= \bar\phi^{(i)} + \alpha_0^{(i)}\ln\lambda + \phi_-^{(i)}(\lambda) + \phi_+^{(i)}(\lambda)\\ &= \bar\phi^{(i)} + \alpha_0^{(i)}\ln\lambda - \sum_{n<0}\frac{\alpha_n^{(i)}}{n}\,\lambda^{-n} - \sum_{n>0}\frac{\alpha_n^{(i)}}{n}\,\lambda^{-n}\quad (i=1,2)\end{aligned} \tag{2.48}$$

with the OPE $\phi^{(i)}(\lambda)\,\phi^{(j)}(\lambda') = \delta^{ij}\,\ln(\lambda-\lambda')$ (equivalently, $[\alpha_m^{(i)},\alpha_n^{(j)}] = m\,\delta^{ij}\,\delta_{m+n,0}$ and $[\alpha_0^{(i)},\bar\phi^{(j)}] = \delta^{ij}$), to bosonize the free fermions as[13]

$$\psi^{(i)}(\lambda) = \sum_{r\in\mathbb{Z}+1/2}\psi_r^{(i)}\lambda^{-r-1/2} = {}^{\circ}_{\circ}\, e^{\phi^{(i)}(\lambda)}\, {}^{\circ}_{\circ}\, K_i, \tag{2.49}$$

$$\bar\psi^{(i)}(\lambda) = \sum_{r\in\mathbb{Z}+1/2}\bar\psi_r^{(i)}\lambda^{-r-1/2} = {}^{\circ}_{\circ}\, e^{-\phi^{(i)}(\lambda)}\, {}^{\circ}_{\circ}\, K_i, \tag{2.50}$$

where K_i is the cocycle, $K_1=1$ and $K_2=(-1)^{\alpha_0^{(1)}}$. Conversely, the bosons are expressed as

$$\partial\phi^{(i)}(\lambda) \equiv {}^{\circ}_{\circ}\,\psi^{(i)}(\lambda)\bar\psi^{(i)}(\lambda)\,{}^{\circ}_{\circ} = \sum_{n\in\mathbb{Z}}\alpha_n^{(i)}\lambda^{-n-1}. \tag{2.51}$$

[13]We define ${}^{\circ}_{\circ}\, e^{\sum_i \beta^{(i)}\phi^{(i)}(\lambda)}\,{}^{\circ}_{\circ} \equiv e^{\sum_i\beta^{(i)}\phi_-^{(i)}(\lambda)}\, e^{\sum_i\bar\phi^{(i)}}\,\lambda^{\sum_i\alpha_0^{(i)}}\, e^{\sum_i\beta^{(i)}\phi_+^{(i)}(\lambda)}$.

Since $\left[\alpha_n^{(i)}, \bar\psi^{(j)}(\lambda)\right] = -\delta^{ij}\lambda^n\,\bar\psi^{(i)}(\lambda)$, the motion (2.46) of a given point in the Grassmannian $\mathcal{M}$ is expressed as[14]

$$\big|\Phi(x)\big\rangle = \rho(x)\, e^{+(1/g)\sum_{n\geq 1}(x_n^{(1)}\alpha_n^{(1)}+x_n^{(2)}\alpha_n^{(2)})}\big|\Phi\big\rangle. \tag{2.52}$$

Here $\big|\Phi\big\rangle \equiv \big|\Phi(0)\big\rangle$ is the initial state at $x=0$. From this state, the Sato-Wilson operator $\boldsymbol{W}(x;\partial)$ can be reconstructed by using the following formula for $\Phi(x;\lambda) = \big(\Phi^{(ij)}(x;\lambda)\big) = \boldsymbol{W}(x;\partial)|_{\partial\to\lambda}$:

$$\Phi^{(ij)}(x;\lambda) = \frac{\big\langle 0\big|e^{-\bar\phi^{(i)}}\psi^{(j)}(\lambda)\big|\Phi(x)\big\rangle}{\big\langle 0\big|\Phi(x)\big\rangle} = \epsilon_{ji}\lambda^{-(1-\delta_{ij})}\,\frac{\big\langle x/g\big|U^{i-j}\,e^{\phi_+^{(j)}(\lambda)}\big|\Phi\big\rangle}{\big\langle x/g\big|\Phi\big\rangle}, \tag{2.53}$$

where $U\equiv e^{\bar\phi^{(1)}-\bar\phi^{(2)}}$, and $\epsilon_{ij}=1$ when $i\leq j$ and $\epsilon_{ij}=-1$ when $i>j$. A proof of this statement can be found, e.g., in [43]. Then the Baker-Akhiezer function is obtained as

$$\begin{aligned}\Psi^{(ij)}(x;\lambda) &= \Phi^{(ij)}(x;\lambda)\cdot\exp\big(g^{-1}\sum_n x_n^{(j)}\lambda^n\big)\\ &= \frac{\big\langle x/g\big|e^{-\bar\phi^{(i)}}\,\psi^{(j)}(\lambda)\big|\Phi\big\rangle}{\big\langle x/g\big|\Phi\big\rangle}.\end{aligned} \tag{2.54}$$

We should note that not every state in the fermion Fock space can be expressed as a point in the Grassmannian $\mathcal{M}$. This correspondence holds if and only if the state $\big|\Phi\big\rangle$ is decomposable, i.e. it can be written as $\big|\Phi\big\rangle = e^H\big|0\big\rangle$ with a fermion bilinear operator H. Since such H is an element of the Lie algebra $gl(\infty)$ realized over the fermion Fock space, the Grassmannian is characterized as an orbit of the infinite dimensional group $GL(\infty)$ [33].

2.5 Background R-R flux

As is investigated in one-matrix models [23 – 25], critical points of 2-cut solutions can have a configuration where the Fermi levels of two Fermi surfaces differ by an integer number ν, which is interpreted as background R-R flux [24, 25]. In the language of 2cKP hierarchy, such configuration is described by a fermion sector with $\alpha_0^{(1)} = -\alpha_0^{(2)} = \nu$ and can be measured by taking the inner product with the state

$$\big\langle x/g;\,\nu\big| \equiv \big\langle x/g\big|\,U^{-\nu}, \tag{2.55}$$

where $U\equiv e^{\bar\phi^{(1)}-\bar\phi^{(2)}}$. Thus, the τ function

$$\tau_\nu(x) \equiv \big\langle x/g;\nu\big|\Phi\big\rangle \tag{2.56}$$

corresponds to the partition function to be obtained in a continuum limit with fixing the discrepancy of the two Fermi levels to be ν. The connected correlation functions with background R-R flux are then given by setting backgrounds as $x=(b_n^{[\mu]})$:

$$\Big\langle \mathcal{O}_{n_1}^{[\mu_1]}\cdots\mathcal{O}_{n_N}^{[\mu_N]}\Big\rangle_{\nu,\mathrm{c}} = \left[\frac{\big\langle b/g;\nu\big|\alpha_{n_1}^{[\mu_1]}\cdots\alpha_{n_N}^{[\mu_N]}\big|\Phi\big\rangle}{\big\langle b/g;\nu\big|\,\Phi\big\rangle}\right]_{\mathrm{c}} \tag{2.57}$$

[14]The factor $\rho(x)$ reflects the fact that the correspondence between a linear space and a fermion state is one-to-one only up to a multiplicative factor.

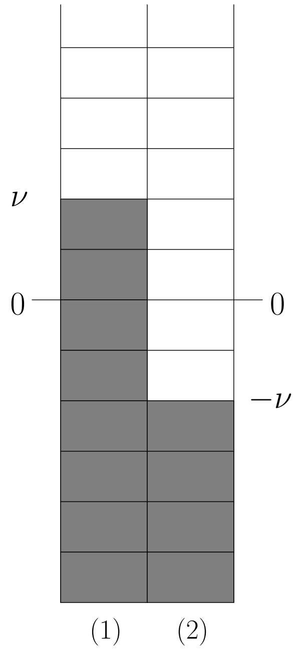

Figure 3: Maya diagram of the state $\big|\nu\big\rangle = U^{\nu}\big|0\big\rangle$.

and have an expansion in the string coupling g as

$$= \sum_{h\geq 0} g^{2h+N-2}\,\Big\langle \mathcal{O}^{[\mu_1]}_{n_1}\cdots\mathcal{O}^{[\mu_N]}_{n_N}\Big\rangle^{(h)}_{\nu,\mathrm{c}}. \tag{2.58}$$

Note that the state $\big\langle x/g;\nu\big|$ is defined in the weak coupling region $|\zeta|\to\infty$, and thus is a state representing a condensate of microscopic-loop operators [46].

The Sato-Wilson operator $\boldsymbol{W}(x;\partial)$ corresponding to the τ function $\tau_\nu(x)$ can be reconstructed by applying the formula (2.53) to $\Phi_\nu(x;\lambda) = \big(\Phi^{(ij)}_\nu(x,\nu;\lambda)\big) = \boldsymbol{W}(x;\partial)|_{\partial\to\lambda}$:

$$\begin{aligned}
\Phi^{(ij)}_\nu(x;\lambda) &= \epsilon_{ji}\lambda^{-(1-\delta_{ij})}\,\frac{\big\langle x/g;\nu\big|U^{i-j}\,e^{\phi^{(j)}_+(\lambda)}\big|\Phi\big\rangle}{\big\langle x/g;\nu\big|\Phi\big\rangle}\\
&= \epsilon_{ji}\lambda^{-(1-\delta_{ij})}\,\frac{1}{\tau_\nu(x)}\,\big\langle x/g;\nu\big|U^{i-j}\,\exp\Big(-\sum_{m\geq 1}\frac{\alpha^{(j)}_m}{m}\lambda^{-m}\Big)\big|\Phi\big\rangle\\
&= \epsilon_{ji}\lambda^{-(1-\delta_{ij})}\,\frac{1}{\tau_\nu(x)}\,\exp\Big(-g\sum_{m\geq 1}\frac{\partial^{(j)}_m}{m}\lambda^{-m}\Big)\big\langle x/g;\nu-i+j\big|\,\Phi\big\rangle\\
&= \epsilon_{ji}\lambda^{-(1-\delta_{ij})}\,\frac{1}{\tau_\nu(x)}\,\sum_{n\geq 0}\lambda^{-n}\,\mathcal{S}_n\big([-g\,\partial^{(j)}_m/m]\big)\,\tau_{\nu-i+j}(x),
\end{aligned} \tag{2.59}$$

where $\mathcal{S}_n([y_n])$ are the Schur polynomials defined by the generating function, $\exp\Big[\sum_{m\geq 1} y_m z^m\Big]$ $= \sum_{n\in\mathbb{Z}}\mathcal{S}_n([y_m])\,z^n$. Expanding the left-hand side as $\Phi_\nu(x;\lambda) = 1 + \sum_{n\geq 1} w_n(x)\,\lambda^{-n}$ and comparing the coefficients of λ^{-n} $(n = 1, 2, \cdots)$, we obtain the formula

$$w_n(x) = \begin{pmatrix} \dfrac{\mathcal{S}_n\big([-g\,\partial_m^{(1)}/m]\big)\,\tau_\nu(x)}{\tau_\nu(x)} & -\dfrac{\mathcal{S}_{n-1}\big([-g\,\partial_m^{(2)}/m]\big)\,\tau_{\nu+1}(x)}{\tau_\nu(x)} \\ \dfrac{\mathcal{S}_{n-1}\big([-g\,\partial_m^{(1)}/m]\big)\,\tau_{\nu-1}(x)}{\tau_\nu(x)} & \dfrac{\mathcal{S}_n\big([-g\,\partial_m^{(2)}/m]\big)\,\tau_\nu(x)}{\tau_\nu(x)} \end{pmatrix}. \tag{2.60}$$

The first two are then given by

$$w_1(x) = \begin{pmatrix} -g\,\partial_1^{(1)}\ln\tau_\nu & -\dfrac{\tau_{\nu+1}}{\tau_\nu} \\ \dfrac{\tau_{\nu-1}}{\tau_\nu} & -g\,\partial_1^{(2)}\ln\tau_\nu \end{pmatrix}, \tag{2.61}$$

$$w_2(x) = \begin{pmatrix} \dfrac{\big(-g\,\partial_2^{(1)} + (g\,\partial_1^{(1)})^2\big)\tau_\nu}{2\tau_\nu} & \dfrac{g\,\partial_1^{(2)}\tau_{\nu+1}}{\tau_\nu} \\ -\dfrac{g\,\partial_1^{(1)}\tau_{\nu-1}}{\tau_\nu} & \dfrac{\big(-g\,\partial_2^{(2)} + (g\,\partial_1^{(2)})^2\big)\tau_\nu}{2\tau_\nu} \end{pmatrix}. \tag{2.62}$$

On the other hand, expanding in ∂ the Sato equations

$$g\,\partial_n^{(i)}\boldsymbol{W}(x;\partial) = -(\boldsymbol{e}^{(i)}\boldsymbol{L}^n)_-\boldsymbol{W}(x;\partial), \tag{2.63}$$

we obtain the relation between $(\boldsymbol{e}^{(i)}\boldsymbol{L}^n)_{-k}(x)$ and $w_n(x)$ $\big($defined by $\big(\boldsymbol{e}^{(i)}\boldsymbol{L}^n\big)(x;\partial) = \sum_{k\in\mathbb{Z}} \big(\boldsymbol{e}^{(i)}\boldsymbol{L}^n\big)_k(x)\,\partial^k$ and $\boldsymbol{W}(x;\partial) = \sum_{n\geq 0} w_n(x)\,\partial^{-n}\big)$:

$$(\boldsymbol{e}^{(i)}\boldsymbol{L}^n)_{-1} = -g\,\partial_n^{(i)}w_1(x), \qquad (\boldsymbol{e}^{(i)}\boldsymbol{L}^n)_{-2} = -g\,\partial_n^{(i)}w_2(x) + g\,\partial_n^{(i)}w_1(x)\cdot w_1(x), \quad \cdots \tag{2.64}$$

Equations (2.60) and (2.64) give the relations which express the coefficients of the pseudo-differential operator $(\boldsymbol{e}^{(i)}\boldsymbol{L}^n)_-$ in terms of τ functions. In particular, looking at the $(1,1)$ component of the relation $(\boldsymbol{e}^{(2)}\boldsymbol{L})_{-1} = -g\,\partial_1^{(2)}w_1(x)$, we obtain the formula

$$g^2\,\partial_1^{(1)}\partial_1^{(2)}\ln\tau_\nu \left(= \frac{g^2}{4}\left[\big(\partial_1^{[0]}\big)^2 - \big(\partial_1^{[1]}\big)^2\right]\ln\tau_\nu(x)\right) = \frac{\tau_{\nu+1}(x)\,\tau_{\nu-1}(x)}{\tau_\nu^2(x)}. \tag{2.65}$$

3. $W_{1+\infty}$ constraints and string equations

In this section, we solve the Douglas equation (2.16), and show that the generating functions (or τ functions) obey the $W_{1+\infty}$ constraints. Using this, we derive string equations in a generic form.

3.1 Solutions to the Douglas equation (2.16)

To solve the Douglas equation (2.16) we first note that $\boldsymbol{P} = \boldsymbol{\sigma}\boldsymbol{L}^{\hat{p}}$ is given by

$$\boldsymbol{P} = \boldsymbol{W}\,\sigma_3\,\partial^{\hat{p}}\,\boldsymbol{W}^{-1}. \tag{3.1}$$

Then the general solution to the Douglas equation (2.16) is given by[15]

$$\boldsymbol{Q} = \boldsymbol{W}\,\frac{1}{\hat{p}}\,\sigma_3\left(\sum_{n\geq 1}\sum_{\mu=0,1} n\,x_n^{[\mu]}\sigma_3^{\mu}\partial^{n-\hat{p}} + g\gamma\,\partial^{-\hat{p}}\right)\boldsymbol{W}^{-1} \tag{3.2}$$

with γ being an arbitrary constant to be fixed later. This can be derived in the way essentially the same with that of the bosonic case ([47, 36], see also Theorem 1 in [9]). Then by requiring that $\boldsymbol{P}$ and $\boldsymbol{Q}$ at the initial time $x = (b_n^{[\mu]})$ be differential operators of order $\hat{p}$ and $\hat{q}$, respectively, we set the background as $b_n^{[\mu]} = 0$ $(n > \hat{p} + \hat{q})$ to obtain:

$$\boldsymbol{P} = \boldsymbol{W}\,\sigma_3\,\partial^{\hat{p}}\,\boldsymbol{W}^{-1} = \left[\boldsymbol{W}\,\sigma_3\,\partial^{\hat{p}}\,\boldsymbol{W}^{-1}\right]_+, \tag{3.3}$$

$$\begin{aligned}\boldsymbol{Q} &= \boldsymbol{W}\,\frac{1}{\hat{p}}\,\sigma_3\left(\sum_{n=1}^{\hat{p}+\hat{q}}\sum_{\mu=0,1} n\,b_n^{[\mu]}\,\sigma_3^{\mu}\partial^{n-\hat{p}} + g\gamma\,\partial^{-p}\right)\boldsymbol{W}^{-1}\\ &= \left[\boldsymbol{W}\,\frac{1}{\hat{p}}\left(\sum_{n=\hat{p}}^{\hat{p}+\hat{q}}\sum_{\mu=0,1} n\,b_n^{[\mu]}\,\sigma_3^{\mu+1}\partial^{n-\hat{p}}\right)\boldsymbol{W}^{-1}\right]_+.\end{aligned} \tag{3.4}$$

Here we have set $\boldsymbol{W} = \boldsymbol{W}(b;\partial)$. Recall that for odd minimal superstrings, we need to set the backgrounds as

$$b_n^{[n+1]} = 0 \quad \text{for odd minimal superstrings: } (p,q) = (\hat{p},\hat{q})\ (\hat{p},\,\hat{q} \in 2\mathbb{Z}+1) \tag{3.5}$$

in order to make $\boldsymbol{Q}$ invariant under the $\mathbb{Z}_2$ transformation $\tilde{\mathrm{C}} : (\boldsymbol{\sigma}, \boldsymbol{L}) \to (-\boldsymbol{\sigma}, -\boldsymbol{L})$.

The action of $\boldsymbol{Q}$ on the Baker-Akhiezer function in a general background is now calculated as follows:

$$\begin{aligned}&\boldsymbol{Q}\,\Psi(x;\lambda) =\\ &= \boldsymbol{W}\frac{1}{\hat{p}}\left(x_1^{[0]}\,\partial^{1-\hat{p}} + x_1^{[1]}\sigma_3\,\partial^{1-\hat{p}} + \sum_{n\geq 2}\sum_{\mu} n\,x_n^{[\mu]}\sigma_3^{\mu}\,\partial^{n-\hat{p}} + g\gamma\,\partial^{-\hat{p}}\right)\sigma_3\,\boldsymbol{W}^{-1}\Psi(x;\lambda)\\ &= \frac{1}{\hat{p}}\left[\boldsymbol{W}, x_1^{[0]}\right]\partial^{1-\hat{p}}\sigma_3\boldsymbol{W}^{-1}\Psi(x;\lambda) + \frac{1}{\hat{p}}\left(\sum_{n\geq 1}\sum_{\mu} n\,x_n^{[\mu]}\boldsymbol{\sigma}^{\mu}\,\boldsymbol{L}^{n-\hat{p}} + g\gamma\,\boldsymbol{L}^{-\hat{p}}\right)\boldsymbol{\sigma}\,\Psi(x;\lambda)\\ &= \left\{\left[\boldsymbol{W}(x;\partial), x_1^{[0]}\right]\cdot\boldsymbol{W}^{-1}\Psi(x;\lambda) + \Psi(x;\lambda)\left(\sum_{n\geq 1}\sum_{\mu} n\,x_n^{\mu}\sigma_3^{\mu}\lambda^{n-1} + g\gamma\lambda^{-1}\right)\right\}\frac{\sigma_3}{\hat{p}}\lambda^{1-\hat{p}}.\end{aligned} \tag{3.6}$$

Since $\left[\boldsymbol{W}, x_1^{[0]}\right] = \left[\sum_{n\geq 0} w_n(x)\partial^{-n}, x_1^{[0]}\right] = g\sum_{n\geq 1}(-n)\,w_n(x)\,\partial^{-n-1} = g\,\boldsymbol{W}(x;\lambda)\,\overleftarrow{\partial_\lambda}\Big|_{\lambda\to\partial}$

[15] For a generic value of (θ_p, θ_q) and ϵ, we have

$$\boldsymbol{P} = \boldsymbol{W}\,\sigma_3^{\epsilon}e^{\theta_p\sigma_3}\partial^{\hat{p}}\,\boldsymbol{W}^{-1}, \qquad \boldsymbol{Q} = \boldsymbol{W}\,\frac{1}{\hat{p}}\,\sigma_3^{\epsilon}e^{-\theta_p\sigma_3}\left(\sum_{n\geq 1}\sum_{\mu=0,1} n\,x_n^{[\mu]}\,\sigma_3^{\mu}\partial^{n-\hat{p}} + g\gamma\,\partial^{-\hat{p}}\right)\boldsymbol{W}^{-1}.$$

$= g\,\Phi(x;\lambda)\,\overleftarrow{\partial_\lambda}$, the above equation can be further rewritten into the following form:

$$\begin{aligned}
&\boldsymbol{Q}\,\Psi(x;\lambda)\\
&= g\left\{\Phi(x;\lambda)\,\overleftarrow{\partial_\lambda}\Big|_{\lambda\to\partial}\cdot\boldsymbol{W}^{-1}\Psi(x;\lambda)+\Phi(x;\lambda)\cdot\left[\left(\boldsymbol{W}^{-1}\Psi(x;\lambda)\right)\frac{\overleftarrow{\partial}}{\partial\lambda}\right]+\gamma\lambda^{-1}\right\}\frac{\sigma_3}{\hat{p}}\,\lambda^{1-\hat{p}}\\
&= g\,\Psi(x;\lambda)\left[\frac{\overleftarrow{\partial}}{\partial\lambda}+\gamma\lambda^{-1}\right]\begin{pmatrix} d\lambda/d\zeta & 0\\ 0 & -d\lambda/d\zeta\end{pmatrix}\qquad(\zeta=\lambda^{\hat{p}}).
\end{aligned}\tag{3.7}$$

Setting $\gamma\equiv-(\hat{p}-1)\nu$, we thus obtain[16]

$$\boldsymbol{P}\,\tilde{\Psi}(x;\zeta)=\tilde{\Psi}(x;\zeta)\begin{pmatrix}\zeta & 0\\ 0 & -\zeta\end{pmatrix},\qquad \boldsymbol{Q}\,\tilde{\Psi}(x;\zeta)=g\,\tilde{\Psi}(x;\zeta)\begin{pmatrix}\frac{\overleftarrow{\partial}}{\partial\zeta} & 0\\ 0 & -\frac{\overleftarrow{\partial}}{\partial\zeta}\end{pmatrix}\tag{3.8}$$

for

$$\tilde{\Psi}(x;\zeta)\equiv\Psi(x;\lambda)\left(\frac{\partial\lambda}{\partial\zeta}\right)^{\nu}.\tag{3.9}$$

It turns out to be convenient to introduce another set of chiral fermions $(c_0^{(i)}(\zeta),\bar{c}_0^{(i)}(\zeta))$ $(i=1,2)$:

$$c_0^{(i)}(\zeta)\equiv\left(\frac{d\lambda}{d\zeta}\right)^{\nu}\psi^{(i)}(\lambda),\qquad \bar{c}_0^{(i)}(\zeta)\equiv\left(\frac{d\lambda}{d\zeta}\right)^{\nu}\bar{\psi}^{(i)}(\lambda).\tag{3.10}$$

By using the representation (2.54), $\tilde{\Psi}(x;\zeta)$ can be written as

$$\tilde{\Psi}^{(ij)}(x;\zeta)=\frac{\langle x/g|e^{-\bar{\phi}^{(i)}}c_0^{(j)}(\zeta)|\Phi\rangle}{\langle x/g|\Phi\rangle}.\tag{3.11}$$

Note that these new fermions have a $\mathbb{Z}_{\hat{p}}$ monodromy, and thus we introduce:

$$c_a^{(i)}(\zeta)\equiv c_0^{(i)}(e^{2\pi ia}\zeta),\quad \bar{c}_a^{(i)}(\zeta)\equiv\bar{c}_0^{(i)}(e^{2\pi ia}\zeta)\quad(a=0,1,\cdots,\hat{p}-1).\tag{3.12}$$

3.2 $W_{1+\infty}$ constraints

The corresponding $W_{1+\infty}$ constraints can be derived also in the way same with that of bosonic case ([47, 36], see also Lemma 1 in [9]), starting from the following expression of the differential operators $\boldsymbol{P}$ and $\boldsymbol{Q}$:

$$\boldsymbol{P}\,\boldsymbol{W}=\boldsymbol{W}\,\sigma_3\,\partial^{\hat{p}},\qquad \boldsymbol{Q}\,\boldsymbol{W}=\boldsymbol{W}\,\sigma_3\left[\sum_{n,\mu}n\,x_n^{[\mu]}\sigma_3^{\mu}\,\partial^{n-\hat{p}}+g\gamma\partial^{-\hat{p}}\right].\tag{3.13}$$

Denoting the initial subspace by $\mathcal{V}\equiv\mathcal{V}(0)=\mathcal{V}(x)\cdot e^{(1/g)\sum_{n,\mu}x_n^{[\mu]}\sigma_3^{\mu}\lambda^n}\subset\mathcal{H}$, one can easily show that

$$\mathcal{P}\equiv\zeta\sigma_3,\qquad \mathcal{Q}\equiv\left[\frac{\overleftarrow{\partial}}{\partial\lambda}\lambda^{1-\hat{p}}+\gamma\lambda^{-\hat{p}}\right]\sigma_3\quad(\gamma=-(\hat{p}-1)\nu)\tag{3.14}$$

[16] Note that this is a representation of the Douglas equation $[\boldsymbol{P},\boldsymbol{Q}]=g\boldsymbol{1}_2$ given by a right multiplication $[\zeta,\overleftarrow{\partial_\zeta}]=1$.

do not leave the vector space $\mathcal{V}$:

$$\mathcal{V}\cdot\mathcal{P}\subset\mathcal{V},\qquad \mathcal{V}\cdot\mathcal{Q}\subset\mathcal{V}. \tag{3.15}$$

Recursively using this invariance, we find that $\mathcal{V}$ is invariant under the right action of $\mathcal{Q}^l\,\mathcal{P}^m$ for arbitrary nonnegative integers l and m,

Since $\mathcal{Q}$ can be written as $\Big[\Big(\frac{d\lambda}{d\zeta}\Big)^{\nu}\frac{\overleftarrow{\partial}}{\partial\zeta}\Big(\frac{d\lambda}{d\zeta}\Big)^{-\nu}\Big]\sigma_3$, the above can be restated for the space of functions of $\zeta=\lambda^{\hat{p}}$ with an extra factor $(d\lambda/d\zeta)^{\nu}$:

$$\tilde{\mathcal{V}}\equiv\mathcal{V}\cdot\Big(\frac{d\lambda}{d\zeta}\Big)^{\nu} \tag{3.16}$$

as

$$\tilde{\mathcal{V}}\cdot\tilde{\mathcal{Q}}^l\,\tilde{\mathcal{P}}^m\subset\tilde{\mathcal{V}}\quad(l,m\geq 0) \tag{3.17}$$

with

$$\tilde{\mathcal{P}}\equiv\zeta\sigma_3,\qquad \tilde{\mathcal{Q}}\equiv\frac{\overleftarrow{\partial}}{\partial\zeta}\sigma_3. \tag{3.18}$$

In term of free fermions, the invariance of the space of functions $\mathcal{V}$ (or $\tilde{\mathcal{V}}$) implies the invariance of the fermion state $\big|\Phi\big\rangle$ under the action of the corresponding bilinear operators

$$\begin{aligned}
O\big[\mathcal{Q}^l\,\mathcal{P}^m\big]&=\oint\frac{d\lambda}{2\pi i}\,{}^{\circ}_{\circ}\,\psi(\lambda)\cdot\mathcal{Q}^l\,\mathcal{P}^m\cdot\bar{\psi}(\lambda)\,{}^{\circ}_{\circ}\\
&=\oint\!\!\!\!\!\bigcirc\,\frac{d\zeta}{2\pi i}\,:c_0(\zeta)\cdot\tilde{\mathcal{Q}}^l\,\tilde{\mathcal{P}}^m\cdot\bar{c}_0(\zeta):\\
&=\oint\!\!\!\!\!\bigcirc\,\frac{d\zeta}{2\pi i}\,:\Big(c_0(\zeta)\frac{\overleftarrow{\partial^l}}{\partial\zeta^l}\Big)\zeta^m(\sigma_3)^{m+l}\bar{c}_0(\zeta):\\
&=\oint\frac{d\zeta}{2\pi i}\sum_{a=0}^{\hat{p}-1}:\Big(c_a(\zeta)\frac{\overleftarrow{\partial^l}}{\partial\zeta^l}\Big)\zeta^m(\sigma_3)^{m+l}\bar{c}_a(\zeta):.
\end{aligned} \tag{3.19}$$

Here the contour of the integral $\oint\!\!\!\!\!\bigcirc$ surrounds $\zeta=\infty$ $\hat{p}$ times. We further bosonize the fields

$$c_a(\zeta)=\big(c_a^{(1)}(\zeta),c_a^{(2)}(\zeta)\big),\qquad \bar{c}_a(\zeta)=\begin{pmatrix}\bar{c}_a^{(1)}(\zeta)\\ \bar{c}_a^{(2)}(\zeta)\end{pmatrix},\qquad (a=0,1,\cdots,\hat{p}-1) \tag{3.20}$$

as[17]

$$c_a^{(i)}(\zeta)=:e^{\varphi_a^{(i)}(\zeta)}:K_a^{(i)},\qquad \bar{c}_a^{(i)}(\zeta)=:e^{-\varphi_a^{(i)}(\zeta)}:K_a^{(i)} \tag{3.21}$$

[17] $K_a^{(i)}$ are proper cocycles which ensure the anticommutation relations among fermion fields with different indices, They can be taken, for example, as $K_a^{(i)}=\prod_{j=1}^{i-1}\prod_{b=0}^{a-1}(-1)^{\alpha_{b,0}^{(j)}}$ with $\alpha_{a,0}^{(i)}$ are the momentum of $\varphi_a^{(i)}(\zeta)$.

with two sets of free chiral bosons

$$\varphi_a^{(i)}(\zeta)\,\varphi_b^{(j)}(\zeta') = +\,\delta^{ij}\,\delta_{ab}\,\ln(\zeta-\zeta') \quad \bigl(i,j=1,2;\; a,b=0,1,\cdots,\hat{p}-1\bigr). \tag{3.22}$$

The normal ordering : : is now the one taken with respect to the $\mathrm{SL}(2,\mathbb{C})$ invariant vacuum for ζ plane, $\bigl|0\bigr\rangle_\zeta$, so that the original vacuum $\bigl|0\bigr\rangle$ (respecting the $\mathrm{SL}(2,\mathbb{C})$ invariance for λ) is interpreted as the twisted vacuum for the chiral bosons living on the ζ plane. The monodromy (3.12) now should be understood as relations to hold in correlation functions with this twisted vacuum (not as operator identities).

By writing $l=s-1$ and $m=s+n-1$ $(s\geq 1;\, n\geq -s+1)$ in (3.19), the operator $O[\mathcal{Q}^l\,\mathcal{P}^m] = O[\mathcal{Q}^{s-1}\,\mathcal{P}^{s+n-1}]$ is then written as[18]

$$\begin{aligned}
&\frac{1}{s}\oint\frac{d\zeta}{2\pi i}\,\zeta^{s+n-1}\sum_a\Bigl[:e^{-\varphi^{(1)}(\zeta)}\partial_\zeta^s e^{\varphi^{(1)}(\zeta)}: +(-1)^n\,:e^{-\varphi^{(2)}(\zeta)}\partial_\zeta^s e^{\varphi^{(2)}(\zeta)}:\Bigr] = \\
&\frac{1}{s}\oint\frac{d\zeta}{2\pi i}\,\zeta^{s+n-1}\sum_a\Bigl[:e^{-\varphi^{(1)}(\zeta)}\partial_\zeta^s e^{\varphi^{(1)}(\zeta)}: +\,:e^{-\varphi^{(2)}(-\zeta)}\partial_\zeta^s e^{\varphi^{(2)}(-\zeta)}:\Bigr].
\end{aligned} \tag{3.23}$$

Thus, if we define the $W_{1+\infty}$ currents $W^s(\zeta)$ $(s=1,2,\cdots)$ as

$$\begin{aligned}
W^s(\zeta) &\equiv \sum_{a=0}^{\hat{p}-1}\mathcal{W}_a^s(\zeta) \\
&= \sum_{a=0}^{\hat{p}-1}\bigl(:e^{-\varphi_a^{(1)}(\zeta)}\partial^s\,e^{\varphi_a^{(1)}(\zeta)}: + :e^{-\varphi_a^{(2)}(-\zeta)}\partial^s\,e^{\varphi_a^{(s)}(-\zeta)}:\bigr) \\
&\equiv \sum_{n\in\mathbb{Z}} W_n^s\,\zeta^{-n-s}
\end{aligned} \tag{3.24}$$

with the coefficient modes W_n^s that naturally constitute the $W_{1+\infty}$ algebra [48], then eq. (3.17) implies the following $W_{1+\infty}$ constraints on the state $\bigl|\Phi\bigr\rangle$

$$W_n^s\bigl|\Phi\bigr\rangle = \text{const.}\bigl|\Phi\bigr\rangle \qquad (s\geq 1;\, n\geq -s+1). \tag{3.25}$$

One can show that any element in the Borel subalgebra spanned by W_n^s with $s\geq 1$ and $n\geq -s+1$ can always be written as a commutator of other two elements belonging to the same subalgebra (see, e.g., Lemma 4.2 of [49]), and thus the constants should vanish. We thus obtain the $W_{1+\infty}$ constraints on the state $\bigl|\Phi\bigr\rangle$:

$$W_n^s\bigl|\Phi\bigr\rangle = 0 \qquad (s\geq 1;\, n\geq -s+1). \tag{3.26}$$

This is equivalent to the statement that the action of the $W_{1+\infty}$ currents on the state $\bigl|\Phi\bigr\rangle$ only yields regular dependence on ζ:

$$W^s(\zeta)\,\bigl|\Phi\bigr\rangle : \text{ regular around } \zeta=0. \tag{3.27}$$

[18]One can show that the generic case of (2.18) can be recovered by properly changing the integration variables ζ separately for $\varphi^{(1)}(\zeta)$ and $\varphi^{(2)}(\zeta)$.

This is the key relation to determine loop amplitudes.

The first two of the $W_{1+\infty}$ currents are given by

$$W^1(\zeta) = \sum_a \big(\partial\varphi_a^{(1)}(\zeta) - \partial\varphi_a^{(2)}(-\zeta)\big), \tag{3.28}$$

$$\begin{aligned} W^2(\zeta) &= \sum_a \Big(:\big(\partial\varphi_a^{(1)}(\zeta)\big)^2: + \partial^2\varphi_a^{(1)}(\zeta) + :\big(-\partial\varphi_a^{(2)}(-\zeta)\big)^2: + \partial^2\varphi_a^{(2)}(-\zeta)\Big) \\ &= \sum_a \Big(\mathring{\circ}\,\big(\partial\varphi_a^{(1)}(\zeta)\big)^2\,\mathring{\circ} + \partial^2\varphi_a^{(1)}(\zeta) + \mathring{\circ}\,\big(-\partial\varphi_a^{(2)}(-\zeta)\big)^2\,\mathring{\circ} + \partial^2\varphi_a^{(2)}(-\zeta)\Big) + \frac{\hat p^2-1}{6\hat p}\frac{1}{\zeta^2} \\ &\equiv 2\sum_{n\in\mathbb{Z}} L_n\,\zeta^{-n-2} + \partial W^1(\zeta). \end{aligned} \tag{3.29}$$

Since the normal ordering with respect to the original vacuum $\big|0\big\rangle$ can be represented in terms of oscillators, we have

$$W^1_n = \alpha^{(1)}_{n\hat p} + (-1)^n\alpha^{(2)}_{n\hat p}, \tag{3.30}$$

$$L_n = \frac{1}{2\hat p}\sum_{n\in\mathbb{Z}}\Big(\mathring{\circ}\,\alpha^{(1)}_{n\hat p-m}\alpha^{(1)}_m\,\mathring{\circ} + (-1)^n\,\mathring{\circ}\,\alpha^{(2)}_{n\hat p-m}\alpha^{(2)}_m\,\mathring{\circ}\Big) + \frac{\hat p^2-1}{12\hat p}\,\delta_{n,0}. \tag{3.31}$$

These imply the p^{th} reduction conditions and the Virasoro constraints on the state (as in the bosonic case [37, 50 − 52]):

$$\big(\alpha^{(1)}_{n\hat p} + (-1)^n\alpha^{(2)}_{n\hat p}\big)\big|\Phi\big\rangle = 0 \quad (n\ge 0), \tag{3.32}$$

$$L_n\big|\Phi\big\rangle = 0 \quad (n\ge -1), \tag{3.33}$$

which are rewritten for the τ functions with background R-R flux ν, $\tau_\nu(x) = \big\langle x/g\big|U^{-\nu}\big|\Phi\big\rangle$, as

$$\big(\partial^{(1)}_{n\hat p} + (-1)^n\partial^{(2)}_{n\hat p}\big)\,\tau_\nu(x) = 0, \qquad \nu^{(1)} = -\nu^{(2)} = \nu, \tag{3.34}$$

and

$$\mathcal{L}_n\,\tau_\nu(x) = 0 \quad (n\ge -1) \tag{3.35}$$

with

$$\begin{aligned} \hat p\mathcal{L}_{+n} &= \frac{g^2}{2}\sum_{m=1}^{n\hat p-1}\big(\partial^{(1)}_{n\hat p-m}\,\partial^{(1)}_m + (-1)^n\partial^{(2)}_{n\hat p-m}\,\partial^{(2)}_m\big) \\ &\quad + \sum_{m\ge1} m\,\big(x^{(1)}_m\,\partial^{(1)}_{m+n\hat p} + (-1)^n\,x^{(2)}_m\,\partial^{(2)}_{m+n\hat p}\big) + g\nu\big(\partial^{(1)}_{n\hat p} - (-1)^n\,\partial^{(2)}_{n\hat p}\big), \end{aligned} \tag{3.36}$$

$$\hat p\mathcal{L}_0 = \sum_{m\ge1} m\,\big(x^{(1)}_m\,\partial^{(1)}_m + x^{(2)}_m\,\partial^{(2)}_m\big) + \frac{\hat p^2-1}{12} + \nu^2, \tag{3.37}$$

$$\begin{aligned} \hat p\mathcal{L}_{-n} &= \frac{1}{2g^2}\sum_{m=1}^{n\hat p-1} m(n\hat p-m)\,\big(x^{(1)}_{n\hat p-m}\,x^{(1)}_m + (-1)^n\,x^{(2)}_{n\hat p-m}\,x^{(2)}_m\big) \\ &\quad + \sum_{m\ge1}(n\hat p+m)\,\big(x^{(1)}_{n\hat p+m}\,\partial^{(1)}_m + (-1)^n\,x^{(2)}_{n\hat p+m}\,\partial^{(2)}_m\big) + \frac{n\hat p}{g}\,\nu\,\big(x^{(1)}_{n\hat p} - (-1)^n\,x^{(2)}_{n\hat p}\big). \end{aligned} \tag{3.38}$$

The differential operators satisfy the Virasoro algebra of central charge $2\hat{p}$, $\big[\mathcal{L}_n, \mathcal{L}_m\big] = (n-m)\,\mathcal{L}_{n+m} + (2\hat{p}\,(n^3-n)/12)\,\delta_{n+m,0}$.

Note that the whole $W_{1+\infty}$ constraints are derived from two of the constraints $W^1_1\tau_\nu = 0$ and $W^2_{-1}\tau_\nu = 0$ (or equivalently, $\mathcal{L}_{-1}\tau_\nu = 0$) if we require that the function τ_ν be a τ function of 2cKP. Thus, for a τ function $\tau_\nu(x)$, the equation

$$\mathcal{L}_{-1}\,\tau_\nu(x) = 0 \tag{3.39}$$

together with the $p^{\rm th}$ reduction conditions (3.34) has the maximal information to determine the partition function with the R-R flux ν.

In terms of pseudo-differential operators, eq. (3.39) is expressed as follows:

$\hat{p}=1$:

$$\begin{aligned} 0 &= \sum_{n\geq 0}(n+1)\Big[x^{(1)}_{n+1}\big(\boldsymbol{e}^{(1)}\boldsymbol{L}^n\big)_{-1} - x^{(2)}_{n+1}\big(\boldsymbol{e}^{(2)}\boldsymbol{L}^n\big)_{-1}\Big] + g\,\nu\,\mathbf{1}_2 \\ &= \sum_{n\geq 0}\sum_{\mu=0,1}(n+1)\,x^{[\mu]}_{n+1}\big(\boldsymbol{\sigma}^{\mu+1}\boldsymbol{L}^n\big)_{-1} + g\,\nu\,\mathbf{1}_2, \end{aligned} \tag{3.40}$$

$\hat{p}=2$:

$$\begin{aligned} 0 &= \sum_{n\geq -1}(n+2)\Big[x^{(1)}_{n+1}\big(\boldsymbol{e}^{(1)}\boldsymbol{L}^n\big)_{-1} - x^{(2)}_{n+1}\big(\boldsymbol{e}^{(2)}\boldsymbol{L}^n\big)_{-1}\Big] \\ &= \sum_{n\geq -1}\sum_{\mu=0,1}(n+2)\,x^{[\mu]}_{n+2}\big(\boldsymbol{\sigma}^{\mu+1}\boldsymbol{L}^n\big)_{-1}, \end{aligned} \tag{3.41}$$

$$\begin{aligned} 0 &= \sum_{n\geq -1}(n+2)\Big[x^{(1)}_{n+2}\big(\boldsymbol{e}^{(1)}\boldsymbol{L}^n\big)_{-2} - x^{(2)}_{n+2}\big(\boldsymbol{e}^{(2)}\boldsymbol{L}^n\big)_{-2}\Big] + g\begin{pmatrix}\nu-1/2 & 0\\ 0 & \nu+1/2\end{pmatrix} \\ &= \sum_{n\geq -1}\sum_{\mu=0,1}(n+2)\,x^{[\mu]}_{n+2}\big(\boldsymbol{\sigma}^{\mu+1}\boldsymbol{L}^n\big)_{-2} + g\begin{pmatrix}\nu-1/2 & 0\\ 0 & \nu+1/2\end{pmatrix}, \end{aligned} \tag{3.42}$$

where

$$\big(\boldsymbol{e}^{(1)}\big)_{-1} = -\big(\boldsymbol{e}^{(2)}\big)_{-1} = \begin{pmatrix}0 & H_+\\ H_- & 0\end{pmatrix}, \tag{3.43}$$

$$\big(\boldsymbol{e}^{(1)}\big)_{-2} = -\big(\boldsymbol{e}^{(2)}\big)_{-2} = -\begin{pmatrix}H_+H_- & g\,\partial^{(2)}_1 H_+\\ g\,\partial^{(1)}_1 H_- & -H_+H_-\end{pmatrix}, \tag{3.44}$$

$$\big(\boldsymbol{e}^{(1)}\boldsymbol{L}^{-1}\big)_{-1} = E^{(1)} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad \big(\boldsymbol{e}^{(2)}\boldsymbol{L}^{-1}\big)_{-1} = E^{(2)} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \tag{3.45}$$

$$\big(\boldsymbol{e}^{(1)}\boldsymbol{L}^{-1}\big)_{-2} = -\big(\boldsymbol{e}^{(2)}\boldsymbol{L}^{-1}\big)_{-2} = \begin{pmatrix}0 & H_+\\ H_- & 0\end{pmatrix} \tag{3.46}$$

with

$$H_\pm(x) \equiv \frac{\tau_{\nu\pm 1}(x)}{\tau_\nu(x)}. \tag{3.47}$$

A proof of the above formulas is given in appendix C.

As an example, we consider the $\hat{p}=1$ case with the differential operator

$$\boldsymbol{P}=\boldsymbol{\sigma}\boldsymbol{L}=\sigma_3\,\partial+2H, \quad H\equiv\begin{pmatrix}0 & H_+\\ H_- & 0\end{pmatrix}. \tag{3.48}$$

The variables $x_n^{[n-1]}$ are only physically important due to the $\hat{p}=1^{\rm st}$ conditions, $\partial_n^{[n]}\tau_\nu(x)=0$. By denoting them by $t_n\equiv x_n^{[n-1]}$, the string equations become

$$\sum_{n=1}^{\hat{q}+1} n\,t_n\,(\boldsymbol{\sigma}\boldsymbol{P}^{n-1})_{-1}+g\,\nu\,\mathbf{1}_2=0. \tag{3.49}$$

The pseudo-differential operator $\boldsymbol{\sigma}$ is easily found by requiring $[\boldsymbol{\sigma},\boldsymbol{P}]=0$ and $\boldsymbol{\sigma}^2=\mathbf{1}_2$, and we obtain

$$\boldsymbol{\sigma}=\sigma_3+2H_1\partial^{-1}+2H_2\partial^{-2}+2H_3\partial^{-3}+\cdots \tag{3.50}$$

with the coefficient functions H_n given in (D.2) of appendix D.

The string equations for the backgrounds $(b_n)=(t_1,0,t_3,0,0,\cdots)$ (purely NS-NS) and $(b_n)=(t_1,0,0,t_4,\cdots)$ (in a flow generated by R-R operator $\mathcal{O}_4^{[1]}$) are given as follows (note that $\partial=g\,\partial/\partial x_1^{[0]}=g\,\partial/\partial t_1$):

$\hat{p}=1,\hat{q}=2$:

$$\begin{cases}0=3t_3(H_-\partial H_+-H_+\partial H_-)+g\,\nu,\\ 0=-2t_1H_+-3t_3\big(\frac{1}{2}\partial^2H_++4H_+^2H_-\big),\\ 0=-2t_1H_--3t_3\big(\frac{1}{2}\partial^2H_-+4H_-^2H_+\big),\end{cases} \tag{3.51}$$

$\hat{p}=1,\hat{q}=3$:

$$\begin{cases}0=4t_4(\frac{1}{2}H_+\partial^2H_-+\frac{1}{2}H_-\partial^2H_+-\frac{1}{2}\partial H_+\partial H_-+6H_+^2H_-^2)+g\,\nu,\\ 0=-2t_1H_+-4t_4\big(\frac{1}{4}\partial^3H_++6H_+H_-\partial H_+\big),\\ 0=-2t_1H_-+4t_4\big(\frac{1}{4}\partial^3H_-+6H_+H_-\partial H_-\big),\end{cases} \tag{3.52}$$

where the equations are displayed in the order of the diagonal, (1,2) and (2,1) elements. Note that the diagonal equation including the flux ν is related with the other equations as

$$\partial(\text{diagonal})=2\times\big[H_-\times(1,2)-H_+\times(2,1)\big], \tag{3.53}$$

from which ν disappears. In fact, R-R flux ν can easily get lost in the analysis other than the Virasoro constraints. For example, instead of using the Virasoro constraints, we could obtain string equations by directly solving the Douglas equation $[\boldsymbol{P},\boldsymbol{Q}]=g\mathbf{1}_2$. This calculation is performed in appendix D, and we find there that the obtained string equations are in the form of the commutator of eq. (3.49) with σ_3 (see eq. (D.8)), so that the information on ν is totally lost because ν appears in the diagonal elements of the Virasoro constraints. For the latter string equations (D.8), the diagonal element is a total derivative and ν is introduced as an integration constant [21, 39, 22, 24, 25]. This is a feature which holds generically for arbitrary $\hat{p}$, and shows the advantage of our string field theoretical approach; the R-R background flux ν has a definite meaning as the discrepancy between two Fermi levels and enters the expression without ambiguity.

3.3 $\mathbb{Z}_2$ symmetries and orbifolding

One can introduce the following Z_2 transformations into the theory:

$$\mathrm{C}: (\boldsymbol{\sigma}, \boldsymbol{L}) \to (-\boldsymbol{\sigma}, \boldsymbol{L}), \tag{3.54}$$

$$\tilde{\mathrm{C}}: (\boldsymbol{\sigma}, \boldsymbol{L}) \to (-\boldsymbol{\sigma}, -\boldsymbol{L}). \tag{3.55}$$

Since the oscillator $\alpha_n^{[\mu]} = \alpha_n^{(1)} + (-1)^\mu \alpha_n^{(2)}$ $(\mu = 0, 1)$ corresponds to the differential operator $(\boldsymbol{\sigma}^\mu \boldsymbol{L}^n)_+$, the above transformations are rewritten in terms of oscillators as[19]

$$\mathrm{C}: \alpha_n^{[\mu]} \to (-1)^\mu\, \alpha_n^{[\mu]}, \qquad \hat{\nu} \to -\hat{\nu} \tag{3.56}$$

$$\tilde{\mathrm{C}}: \alpha_n^{[\mu]} \to (-1)^{n+\mu}\, \alpha_n^{[\mu]}, \quad \hat{\nu} \to -\hat{\nu} \tag{3.57}$$

or

$$\mathrm{C}: \alpha_n^{(1)} \leftrightarrow \alpha_n^{(2)}, \tag{3.58}$$

$$\tilde{\mathrm{C}}: \alpha_n^{(1)} \leftrightarrow (-1)^n\, \alpha_n^{(2)}. \tag{3.59}$$

Looking at the definition (3.24) of the $W_{1+\infty}$ currents and noting that the generators have the following form in the twisted vacuum:

$$W_n^s = \sum_{r=0}^{s} \sum_{m_1+\cdots+m_r = n\hat{p}} w_{m_1\cdots m_r} \left({}^\circ_\circ\, \alpha_{m_1}^{(1)} \cdots \alpha_{m_r}^{(1)} \,{}^\circ_\circ + (-1)^n \,{}^\circ_\circ\, \alpha_{m_1}^{(2)} \cdots \alpha_{m_r}^{(2)} \,{}^\circ_\circ \right), \tag{3.60}$$

we find that under these $\mathbb{Z}_2$ transformations the generators simply transform with multiplicative factors:

$$\mathrm{C}: W_n^s \to (-1)^n\, W_n^s, \tag{3.61}$$

$$\tilde{\mathrm{C}}: W_n^s \to (-1)^{n(\hat{p}+1)}\, W_n^s. \tag{3.62}$$

Thus, if a state $\big|\Phi\big\rangle$ satisfies the $W_{1+\infty}$ constraints, so does the transformed state $\big|\Phi'\big\rangle = \mathrm{C}\big|\Phi\big\rangle$ or $\tilde{\mathrm{C}}\big|\Phi\big\rangle$. Assuming the perturbative uniqueness of solutions to the $W_{1+\infty}$ constraints, we conclude that $\big|\Phi\big\rangle$ and $\big|\Phi'\big\rangle$ can differ only by a multiplication of D-instanton operators and we have the following perturbative identity for any operator $\mathcal{O}$:

$$\frac{\big\langle b/g;\nu\big|\mathcal{O}\big|\Phi\big\rangle}{\big\langle b/g;\nu\big|\Phi\big\rangle} \left(= \frac{\big\langle b'/g;\nu'\big|\mathcal{O}'\big|\Phi'\big\rangle}{\big\langle b'/g;\nu'\big|\Phi'\big\rangle} \right) \simeq \frac{\big\langle b'/g;\nu'\big|\mathcal{O}'\big|\Phi\big\rangle}{\big\langle b'/g;\nu'\big|\Phi\big\rangle}. \tag{3.63}$$

Here $b' = \big(b_n'^{(i)}\big)$ and $\mathcal{O}'$ are the transforms of $b = \big(b_n^{(i)}\big)$ and $\mathcal{O}$, respectively, under the transformation C or $\tilde{\mathrm{C}}$. In particular, if the backgrounds are invariant under one of such transformations, then the expectation values vanish perturbatively for those operators that are not invariant under the transformation. Note that the purely NS-NS backgrounds where $b_n^{(1)} = b_n^{(2)}$ (or $b_n^{[1]} = 0$) are invariant under the charge conjugation C, so that the correlation functions vanish perturbatively if they contain odd number of R-R operators.

[19] The charge conjugation in terms of infinite Grassmannian is described in appendix B.

We introduce the orbifolding with the $\mathbb{Z}_2$ transformation C (*resp.* $\tilde{\mathrm{C}}$) as the following condition on a decomposable fermion state satisfying the $W_{1+\infty}$ constraints:

$$\alpha_n^{[1]}\big|\Phi\big\rangle = 0 \quad (\text{for C}), \qquad \alpha_n^{[n+1]}\big|\Phi\big\rangle = 0 \quad (\text{for } \tilde{\mathrm{C}}), \tag{3.64}$$

which is preserved along the 2cKP flows generated by $\alpha_n^{[0]}$ (*resp.* $\alpha_n^{[n]}$) alone. Under this condition, all the correlation functions including $\alpha_n^{[1]}$ (*resp.* $\alpha_n^{[n+1]}$) vanish, and thus these operators drop out of the spectrum. In particular, type 0B theory with the C orbifolding is identified with type 0A theory, while theories of $\hat{p}$, $\hat{q} \in 2\mathbb{Z}+1$ with the $\tilde{\mathrm{C}}$ orbifolding are identified with $(p,q) = (\hat{p},\hat{q})$ minimal superstrings.

4. (p,q) minimal superstring field theory

In this section, we construct field theory of (p,q) minimal superstrings.

4.1 String fields and $\eta = -1$ FZZT branes

According to our ansatz on operator identification given in subsection 2.3, the excitations in the NS-NS and R-R sectors are collected into the NS-NS and R-R scalars given by

$$\underline{\text{NS-NS}}: \quad \partial\varphi_0^{[0]}(\zeta) = \partial\varphi_0^{(1)}(\zeta) + \partial\varphi_0^{(2)}(\zeta) = \frac{1}{\hat{p}} \sum_{n\in\mathbb{Z}} \alpha_n^{[0]}\, \zeta^{-n/\hat{p}-1}, \tag{4.1}$$

$$\underline{\text{R-R}}: \quad \partial\varphi_0^{[1]}(\zeta) = \partial\varphi_0^{(1)}(\zeta) - \partial\varphi_0^{(2)}(\zeta) = \frac{1}{\hat{p}} \sum_{n\in\mathbb{Z}} \alpha_n^{[1]}\, \zeta^{-n/\hat{p}-1}. \tag{4.2}$$

Their connected correlation functions (or cumulants) in the presence of R-R flux ν are given by

$$\Big\langle \partial\varphi_0^{(i_1)}(\zeta_1)\cdots\partial\varphi_0^{(i_N)}(\zeta_N)\Big\rangle_{\nu,\mathrm{c}} = \left[\frac{\big\langle b/g;\nu\big|\, :\partial\varphi_0^{(i_1)}(\zeta_1)\cdots\partial\varphi_0^{(i_N)}(\zeta_N):\,\big|\Phi\big\rangle}{\big\langle b/g;\nu\big|\,\Phi\big\rangle}\right]_{\mathrm{c}} \tag{4.3}$$

and have an expansion in the string coupling g as

$$= \sum_{h\geq 0} g^{2h+N-2} \Big\langle\, \partial\varphi_0^{(i_1)}(\zeta_1)\cdots\partial\varphi_0^{(i_N)}(\zeta_N)\,\Big\rangle_{\nu,\mathrm{c}}^{(h)}. \tag{4.4}$$

As in the bosonic case [5], the disk ($h=0$, $N=1$) and annulus ($h=0$, $N=2$) amplitudes have irregular terms which cannot be expressed with the local operators $\mathcal{O}_n^{(i)}$:

$$\Big\langle \mathcal{O}^{(i)}(\zeta)\Big\rangle_{\nu} = \frac{1}{\hat{p}} \sum_{n=1}^{\infty} \Big\langle \mathcal{O}_n^{(i)}\Big\rangle_{\nu}\, \zeta^{-n/\hat{p}-1} + g^{-1}\, N_1^{(i)}(\zeta) \tag{4.5}$$

$$\Big\langle \mathcal{O}^{(i_1)}(\zeta_1)\,\mathcal{O}^{(i_2)}(\zeta_2)\Big\rangle_{\nu,\mathrm{c}} = \frac{1}{\hat{p}^2} \sum_{n_1,n_2=1}^{\infty} \Big\langle \mathcal{O}_{n_1}^{(i_1)}\,\mathcal{O}_{n_2}^{(i_2)}\Big\rangle_{\nu,\mathrm{c}}\, \zeta_1^{-n_1/\hat{p}-1}\zeta_2^{-n_2/\hat{p}-1} + g^0\, N_2^{(i_1 i_2)}(\zeta_1,\zeta_2) \tag{4.6}$$

$$\Big\langle \mathcal{O}^{(i_1)}(\zeta_1)\cdots\mathcal{O}^{(i_N)}(\zeta_N)\Big\rangle_{\nu,\mathrm{c}} = \frac{1}{\hat{p}^N} \sum_{n_1,\cdots n_N=1}^{\infty} \Big\langle \mathcal{O}_{n_1}^{(i_1)}\cdots\mathcal{O}_{n_N}^{(i_N)}\Big\rangle_{\nu,\mathrm{c}}\, \zeta_1^{-n_1/\hat{p}-1}\cdots\zeta_N^{-n_N/\hat{p}-1} \ (N\geq 3). \tag{4.7}$$

These terms (sometimes called "nonuniversal terms" though they are actually universal) are calculated in matrix models and found to be

$$N_1^{(i)}(\zeta) = \frac{1}{\hat{p}} \sum_{n=1}^{\hat{q}+\hat{p}} n\, b_n^{(i)}\, \zeta^{n/\hat{p}-1} \tag{4.8}$$

$$\begin{aligned} N_2^{(i_1 i_2)}(\zeta_1,\zeta_2) &= \delta^{i_1 i_2} \frac{\partial}{\partial\zeta_1}\frac{\partial}{\partial\zeta_2}\big[\ln\big(\lambda_1-\lambda_2\big)-\ln\big(\zeta_1-\zeta_2\big)\big] \\ &= \delta^{i_1 i_2}\Big[\frac{d\lambda_1}{d\zeta_1}\frac{d\lambda_2}{d\zeta_2}\frac{1}{(\lambda_1-\lambda_2)^2}-\frac{1}{(\zeta_1-\zeta_2)^2}\Big] \qquad (\lambda=\zeta^{1/\hat{p}}). \end{aligned} \tag{4.9}$$

Note that they do not depend on ν.

By looking at the algebraic curves defined by disk amplitudes, we will see in the next section that $\varphi^{(i)}(\zeta)$ $(i=1,2)$ correspond to the boundary states of $\eta=-1$ charged FZZT branes:

$$\underline{\text{boundary states}}: \qquad \big|\mathrm{FZZT}+;\zeta\big\rangle = \varphi_0^{(1)}(\zeta), \quad \big|\mathrm{FZZT}-;-\zeta\big\rangle = \varphi_0^{(2)}(-\zeta). \tag{4.10}$$

Once a charged FZZT brane is located at a point in spacetime with coordinate ζ^2, it becomes a source of fundamental strings, with a bunch of worldsheets which are not connected with each other in the sense of worldsheet topology, but are connected in spacetime with their boundaries pinched at the same superspace point ζ. These configurations are easily summed up to give an exponential form as in [7], realizing the spacetime combinatorics of Polchinski [53]:

$$\underline{\text{charged FZZT branes}}: \ \ c_a^{(1)}(\zeta) = :e^{\varphi_a^{(1)}(\zeta)}:, \quad c_a^{(2)}(-\zeta) = :e^{\varphi_a^{(2)}(-\zeta)}: \quad (a=0,1,\cdots,\hat{p}-1). \tag{4.11}$$

With these operators, the R-R charge operator $\hat{\nu} = (1/2)\big(\alpha_0^{(1)}-\alpha_0^{(2)}\big) = (1/2\hat{p})\sum_a\big(\alpha_{a,0}^{(1)}-\alpha_{a,0}^{(2)}\big)$ have the commutation relations

$$\big[\hat{\nu}, c_a^{(1)}(\zeta)\big] = \frac{1}{2}\, c_a^{(1)}(\zeta), \quad \big[\hat{\nu}, c_a^{(2)}(-\zeta)\big] = -\frac{1}{2}\, c_a^{(2)}(-\zeta), \tag{4.12}$$

and thus, the generic partition functions in the presence of R-R flux ν are given by

$$B_\nu\big(\zeta_1^{(1)},\cdots,\zeta_{N_1}^{(1)};\zeta_1^{(2)},\cdots,\zeta_{N_2}^{(2)}\big) \equiv \frac{\Big\langle b/g;\,\nu+\frac{N_1-N_2}{2}\Big|\ :\prod_{k=1}^{N_1} c_{a_k}^{(1)}(\zeta_k^{(1)})\cdot\prod_{l=1}^{N_2} c_{b_l}^{(2)}(-\zeta_l^{(2)}):\ \Big|\Phi\Big\rangle}{\big\langle b/g;\,\nu\big|\Phi\big\rangle} \tag{4.13}$$

with

$$\big\langle b/g;\,\nu+(N_1-N_2)/2\big| \equiv \big\langle b/g;\,\nu\big|\, e^{-N_1\bar{\phi}^{(1)}-N_2\bar{\phi}^{(2)}}. \tag{4.14}$$

All the operators introduced above have bosonized forms, and thus their correlation functions can be calculated (at least perturbatively) once the correlation functions $\big\langle\mathcal{O}^{(i_1)}(\zeta_1)\,\cdots\,\mathcal{O}^{(i_N)}(\zeta_N)\big\rangle_{\nu,\mathrm{c}}$ are obtained.

4.2 D-instanton operators and $\eta = -1$ ZZ branes

As in the bosonic case [5], one can easily show that the commutator of the operator $c_a^{(i)}(\zeta^{(i)})\,\bar{c}_b^{(j)}(\zeta^{(j)})$ ($\zeta^{(1)} \equiv \zeta$, $\zeta^{(2)} \equiv -\zeta$) with the generators of the $W_{1+\infty}$ algebra gives total derivatives unless $i = j$ and $a = b$, and thus the D-instanton operators [5]

$$D_{ab}^{(ij)} = \epsilon_{ji} \oint \frac{d\zeta}{2\pi i}\, c_a^{(i)}(\zeta^{(i)})\bar{c}_b^{(j)}(\zeta^{(j)}) = \oint \frac{d\zeta}{2\pi i}\, :e^{\varphi_a^{(i)}(\zeta^{(i)})-\varphi_b^{(j)}(\zeta^{(j)})}: \\ (i = j \text{ with } a \neq b;\ i \neq j \text{ with } \forall(a,b)) \tag{4.15}$$

commute with the generators:

$$\big[D_{ab}^{(ij)},\, W_n^s\big] = 0. \tag{4.16}$$

This implies that for a given solution $\big|\Phi\big\rangle$ to the $W_{1+\infty}$ constraints, any product of D-instanton operators is again a solution. In order to keep the property that the resulting state is decomposable (i.e. can be written as $g\big|0\big\rangle$ with g an exponential of fermion bilinears), the only possible form of such state is given by

$$\big|\Phi;\theta\big\rangle \equiv \prod_{i,j}\prod_{a,b} e^{\theta_{ab}^{(ij)}\, D_{ab}^{(ij)}}\big|\Phi\big\rangle \tag{4.17}$$

with constants $\theta_{ab}^{(ij)}$ [6]. Note that we need to omit the case $i = j$ and $a = b$. Since $D_{ab}^{(ij)}$ includes the operator $e^{\bar{\phi}^{(i)}-\bar{\phi}^{(j)}}$, it changes the fermion number $(\nu^{(1)},\nu^{(2)})$ when $i \neq j$. We call $D_{ab}^{(11)}$ and $D_{ab}^{(22)}$ the neutral D-instanton operators and $D_{ab}^{(12)}$ and $D_{ab}^{(21)}$ the charged D-instanton operators.

We here write down their explicit form:

neutral D-instanton operators ($a \neq b$):

$$D_{ab}^{(11)} = \oint \frac{d\zeta}{2\pi i}\, c_a^{(1)}(\zeta)\,\bar{c}_b^{(1)}(\zeta) = \oint \frac{d\zeta}{2\pi i}\, :e^{\varphi_a^{(1)}(\zeta)-\varphi_b^{(1)}(\zeta)}:, \tag{4.18}$$

$$D_{ab}^{(22)} = \oint \frac{d\zeta}{2\pi i}\, c_a^{(2)}(-\zeta)\,\bar{c}_b^{(2)}(-\zeta) = \oint \frac{d\zeta}{2\pi i}\, :e^{\varphi_a^{(2)}(-\zeta)-\varphi_b^{(2)}(-\zeta)}: . \tag{4.19}$$

charged D-instanton operators ($\forall a, \forall b$):

$$D_{ab}^{(12)} = -\oint \frac{d\zeta}{2\pi i}\, c_a^{(1)}(\zeta)\,\bar{c}_b^{(2)}(-\zeta) = \oint \frac{d\zeta}{2\pi i}\, :e^{\varphi_a^{(1)}(\zeta)-\varphi_b^{(2)}(-\zeta)}:, \tag{4.20}$$

$$D_{ab}^{(21)} = \oint \frac{d\zeta}{2\pi i}\, c_a^{(2)}(-\zeta)\,\bar{c}_b^{(1)}(\zeta) = \oint \frac{d\zeta}{2\pi i}\, :e^{\varphi_a^{(2)}(-\zeta)-\varphi_b^{(1)}(\zeta)}: . \tag{4.21}$$

When we insert a D-instanton operator $D_{ab}^{(ij)}$ into correlation functions and take the weak coupling limit $g \to +0$, the operator behaves as

$$\langle D_{ab}^{(ij)}\rangle = \oint \frac{d\zeta}{2\pi i}\, e^{(1/g)\,\Gamma_{ab}^{(ij)}(\zeta)+O(g^0)} \tag{4.22}$$

with[20]

$$\Gamma^{(ij)}_{ab}(\zeta) = \big\langle \varphi^{(i)}_a(\zeta^{(i)})\big\rangle^{(h=0)} - \big\langle \varphi^{(j)}_b(\zeta^{(j)})\big\rangle^{(h=0)}. \tag{4.23}$$

If the integral has a saddle point $\zeta = \zeta^*$ and the contour can be deformed in such a way that it passes the saddle point in the standard direction and also that $\Gamma^{(ij)}_{ab}(\zeta)$ takes negative values all along the contour, then the presence of the D-instanton operator gives a controllable, finite nonperturbative effect, which can be evaluated in the vicinity of the saddle point [5]. If we take a conformal background (defined later), the $\Gamma^{(ij)}_{ab}(\zeta)$ at saddle points give the partition functions of stable ZZ branes [8, 9].

5. Algebraic curves for $\eta = -1$ FZZT branes

The algebraic curves defined by the FZZT branes $\varphi^{(i)}_0(\zeta)$ $(i = 1, 2)$ can be calculated from the $W_{1+\infty}$ constraints with proper boundary conditions supplied by the structure of 2cKP hierarchy. This procedure is totally the same with the one for the bosonic case [9].

We first note that the disk amplitudes

$$\begin{aligned} Q^{(i)}(\zeta) &\equiv \lim_{g\to 0} g\, \frac{\big\langle b/g;\, \nu\big|\, \partial\varphi^{(i)}_0(\zeta)\, \big|\Phi\big\rangle}{\big\langle b/g;\, \nu\big|\, \Phi\big\rangle} \\ &= \frac{1}{\hat{p}} \sum_{n=1}^{\hat{q}+\hat{p}} n\, b^{(i)}_n\, \zeta^{n/\hat{p}-1} + \frac{1}{\hat{p}} \sum_{n=1}^{\infty} v^{(i)}_n\, \zeta^{-n/\hat{p}-1}, \end{aligned} \tag{5.1}$$

do not depend on ν when $\nu^{(1)} = -\nu^{(2)} = \nu$ is finite. Here $v^{(i)}_n \equiv \lim_{g\to 0} g\, \big\langle \alpha^{(i)}_n \big\rangle$ is the expectation value of the operator $\alpha^{(i)}_n$ on sphere. In fact, as for the first term it should be clear that it does not depend on ν. As for the second term, the $v^{(i)}_n$ is the expectation value taken for the background $\big\langle b/g;\, \nu\big|$, and thus has a potential dependence on ν. However, the expectation values taken for $\big\langle b/g;\, \nu\big|$ and for $\big\langle b/g;\, \nu = 0\big| \equiv \big\langle b/g\big|$ differ only by a next leading term, so that we can safely neglect the presence of ν in calculating disk amplitudes. We comment that if the R-R flux $\nu^{(i)}$ is tuned as $\nu^{(i)}(g) = \tilde{\nu}^{(i)}/g$, then an extra term $\tilde{\nu}^{(i)}/\hat{p}\,\zeta$ appears in the last expression of (5.1).

The master field equation to obtain disk amplitudes is given by the fact that the expectation values of the $W_{1+\infty}$ currents are polynomials in ζ (see (3.27)):

$$\big\langle W^s(\zeta)\big\rangle_\nu \left(= \frac{\big\langle b/g;\, \nu\big|\, W^s(\zeta)\, \big|\Phi\big\rangle}{\big\langle b/g;\, \nu\big|\Phi\big\rangle} \right) \equiv s\, a_s(\zeta)/g^s. \tag{5.2}$$

Decomposing the $W_{1+\infty}$ currents as a sum over $a = 0, 1, \cdots, \hat{p}-1$ as in (3.24), the above equation is rewritten as (see [9])[21]

$$\sum_{a=0}^{\hat{p}-1} \big\langle \mathcal{W}^s_a(\zeta)\big\rangle_\nu = \hat{p}\, \big[\, \big\langle \mathcal{W}^s_0(\zeta)\big\rangle_\nu \big]_{\rm pol} = s\, a_s(\zeta)/g^s. \tag{5.3}$$

[20] We see in the next section that the disk amplitudes do not depend on the R-R flux ν.

[21] Here for a series expansion of a function $f(\zeta) = \sum_{n\in\mathbb{Z}} f_n\, \zeta^{n/\hat{p}}$, we define its polynomial part as $\big[f(\zeta)\big]_{\rm pol} \equiv \sum_{m\geq 0} f_{m\hat{p}}\, \zeta^m$.

In the weak coupling limit, $\langle \mathcal{W}_s^a(\zeta)\rangle$ receives the dominant contributions from the products of one-point functions with maximal number (equal to s), and thus we have

$$\lim_{g\to 0} g^s \,\langle \mathcal{W}_s^a(\zeta)\rangle = \sum_{a=0}^{\hat{p}-1}\Big[\big(Q_a^{(1)}(\zeta)\big)^s + \big(-Q_a^{(2)}(-\zeta)\big)^s\Big] = s\, a_s(\zeta). \tag{5.4}$$

Here we have introduced

$$Q_a^{(i)}(\zeta) \equiv \Big\langle \partial\varphi_a^{(i)}(\zeta)\Big\rangle^{(0)} \qquad (i=1,2). \tag{5.5}$$

The algebraic curve is then defined by

$$F(\zeta,Q) \equiv \prod_{a=0}^{\hat{p}-1}\big(Q - Q_a^{(1)}(\zeta)\big)\cdot \prod_{a=0}^{\hat{p}-1}\big(Q + Q_a^{(2)}(-\zeta)\big) = 0. \tag{5.6}$$

This is actually a polynomial both of ζ and Q. In fact, if we set $2\hat{p}$ variables as

$$Q_r(\zeta) \equiv \begin{cases} Q_a^{(1)}(\zeta) & (r = a = 0,1,\cdots,\hat{p}-1), \\ -Q_a^{(2)}(-\zeta) & (r = \hat{p}+a = \hat{p},\, \hat{p}+1,\cdots,2\hat{p}-1), \end{cases} \tag{5.7}$$

then $F(\zeta,Q)$ is expanded in Q with the coefficients which are polynomials of the variables $a_s(\zeta)$:

$$\begin{aligned} F(\zeta,Q) &= \prod_{r=0}^{2\hat{p}-1}\big(Q - Q_r(\zeta)\big) = Q^{2\hat{p}}\cdot \exp\Bigg[\sum_{r=0}^{2\hat{p}-1} \ln\Big(1 - \frac{Q_r(\zeta)}{Q(\zeta)}\Big)\Bigg] \\ &= Q^{2\hat{p}}\cdot \exp\Big[-\sum_{s\geq 1} a_s(\zeta)\, Q^{-s}\Big] \\ &= \sum_{k=0}^{2\hat{p}} \mathcal{S}_k\big([-a_s(\zeta)]\big)\, Q^{2\hat{p}-s}. \end{aligned} \tag{5.8}$$

The curve (5.6) has a solution $Q = Q_0^{(1)}(\zeta)$, and thus describes the FZZT branes carrying positive charge. The curve for negative charge must have a solution $Q = Q^{(2)}(\zeta)$ and thus is given by the curve

$$F^{\mathrm{C}}(\zeta,Q) = \prod_{a=0}^{\hat{p}-1}(Q - Q_a^{(2)}(\zeta))\cdot \prod_{a=0}^{\hat{p}-1}(Q + Q_a^{(1)}(-\zeta)) = F(-\zeta,-Q). \tag{5.9}$$

Note that the curve has a dependence on the backgrounds $b=(b_n)$, $F(\zeta,Q;b)=0$, and the curve for negative charge branes are obtained by replacing the backgrounds in the curve for positive charge branes with their charge conjugates:

$$F^{\mathrm{C}}(\zeta,Q;b)\,\Big(= F(-\zeta,-Q;b)\Big) = F(\zeta,Q;b^{\mathrm{C}}). \tag{5.10}$$

In fact, using in the perturbative equality (3.63) the fact that the charge conjugation exchanges $c_a^{(1)}(\zeta)$ and $c_a^{(2)}(\zeta)$ and that ν can be neglected for one-point functions $Q^{(i)}(\zeta)$, we have

$$\begin{aligned} Q_a^{(1)}(\zeta;b) &= \lim_{g\to 0} g\,\frac{\langle b/g|\,\partial\varphi_a^{(1)}(\zeta)\,|\Phi\rangle}{\langle b/g|\Phi\rangle} = \lim_{g\to 0} g\,\frac{\langle b^{\mathrm{C}}/g|\,\partial\varphi_a^{(2)}(\zeta)\,|\Phi^{\mathrm{C}}\rangle}{\langle b^{\mathrm{C}}/g|\Phi^{\mathrm{C}}\rangle} \\ &= \lim_{g\to 0} g\,\frac{\langle b^{\mathrm{C}}/g|\,\partial\varphi_a^{(2)}(\zeta)\,|\Phi\rangle}{\langle b^{\mathrm{C}}/g|\Phi\rangle} = Q_a^{(2)}(\zeta;b^{\mathrm{C}}), \end{aligned} \tag{5.11}$$

so that we obtain the relation, $a_s(\zeta;b) = (-1)^s\,a_s(-\zeta;b^{\mathrm{C}})$, which proves (5.10).

5.1 1-cut and 2-cut solutions

Generically the functions $a_s(\zeta)$ become polynomials for all $s\geq 1$ only when we add the two terms in (5.4), giving general 2-cut solutions. However, for some particular states $|\Phi\rangle$ it happens that each of the two gives a polynomial separately:

$$\sum_{a=0}^{\hat p-1}\big(Q_a^{(1)}(\zeta)\big)^s = s\,a_s^{(1)}(\zeta),\quad \sum_{a=0}^{\hat p-1}\big(Q_a^{(2)}(\zeta)\big)^s = s\,a_s^{(2)}(\zeta)\quad (\forall s\geq 1). \tag{5.12}$$

This happens in the case when the state is factorized as

$$|\Phi\rangle = |\Phi^{(1)}\rangle\otimes|\Phi^{(2)}\rangle = g^{(1)}\otimes g^{(2)}\,|0\rangle \tag{5.13}$$

with $|\Phi^{(i)}\rangle$ constructed only with $\phi^{(i)}(\lambda)$ and satisfying the $W_{1+\infty}$ constraints independently:

$$(W^{(1)})^s_n|\Phi\rangle = (W^{(2)})^s_n|\Phi\rangle = 0 \qquad (s\geq 1;\ n\geq -s+1), \tag{5.14}$$

where $(W^{(i)})^s(\zeta) = \sum_n (W^{(i)})^s_n\,\zeta^{-n-s} \equiv \sum_{a=0}^{\hat p-1} :e^{-\varphi_a^{(i)}(\zeta)}\partial^s e^{\varphi_a^{(i)}(\zeta)}:$. In this case, the algebraic equation is given by the product of polynomials:

$$F(\zeta,Q) = (-1)^{\hat p}\,F^{(1)}(\zeta,Q)\cdot F^{(2)}(-\zeta,-Q) = 0. \tag{5.15}$$

The solution is thus expressed by the algebraic curve $F^{(1)}(\zeta,Q)=0$, and we call this a 1-cut solution. Actually, in order to realize the situation (5.12) alone, we can multiply the state (5.13) with D-instanton operators since their nonperturbative contributions do not appear in the algebraic curve. Since neutral D-instanton operators can be absorbed into $|\Phi^{(1)}\rangle$ or $|\Phi^{(2)}\rangle$, we only need to consider the charged D-instanton operators. We thus obtain the general form of a state corresponding to 1-cut solutions:

$$|\Phi\rangle = \prod_{a,b} e^{\theta_{ab}^{(12)}\,D_{ab}^{(12)}+\theta_{ab}^{(21)}\,D_{ab}^{(21)}}\,|\Phi^{(1)}\rangle\otimes|\Phi^{(2)}\rangle. \tag{5.16}$$

Note that once we take into account the nonperturbative effects from the charged D-instanton operators, the functions $a_s^{(1)}(\zeta)$ and $a_s^{(2)}(\zeta)$ are no longer polynomials of ζ separately, and the disk amplitudes are better described by the curves of 2-cut solutions.

We comment that only 1-cut solutions are allowed for such backgrounds that have both of C and $\tilde{\mathrm{C}}$ symmetries. Recall that we impose the $\tilde{\mathrm{C}}$ invariance only on odd minimal superstrings, $(p,q)=(\hat{p},\hat{q})$ with $\hat{p},\,\hat{q}\in 2\mathbb{Z}+1$. To prove the statement, it is sufficient to show the following equality (shown at the end of this subsection):

$$\sum_a \big(-Q_a^{(2)}(-\zeta)\big)^s = \sum_a \big(Q_a^{(1)}(\zeta)\big)^s, \tag{5.17}$$

which together with (5.4) implies that $a_s^{(1)}(\zeta)=(-1)^s a_s^{(2)}(-\zeta)=(1/2)a_s(\zeta)$ (polynomial). Furthermore, the second factor in the algebraic equation (5.15) can be rewritten as

$$\begin{aligned}
F^{(2)}(-\zeta,-Q) &= \sum_{k=0}^{\hat{p}-k} \mathcal{S}_k([-a_s^{(2)}(-\zeta)])\,(-Q)^{\hat{p}-k} \\
&= \sum_{k=0}^{\hat{p}-k} \mathcal{S}_k([-(-1)^s\, a_s^{(1)}(\zeta)])\,(-Q)^{\hat{p}-k} \\
&= (-1)^{\hat{p}} \sum_{k=0}^{\hat{p}-k} \mathcal{S}_k([-\,a_s^{(1)}(\zeta)])\,Q^{\hat{p}-k} \\
&= (-1)^{\hat{p}}\,F^{(1)}(\zeta,Q),
\end{aligned} \tag{5.18}$$

so that the algebraic equation has the complete square form:

$$F(\zeta,Q)=\big(F^{(1)}(\zeta,Q)\big)^2=0 \quad \text{(for C- and $\tilde{\mathrm{C}}$-invariant backgrounds).} \tag{5.19}$$

We conclude this subsection with showing eq. (5.17) for C- and $\tilde{\mathrm{C}}$-invariant backgrounds. We first rewrite the left-hand side as

$$\begin{aligned}
\sum_a \big(-Q_a^{(2)}(-\zeta)\big)^s &= \sum_a \big(-Q_a^{(1)}(-\zeta)\big)^s \quad \text{(C invariance)} \\
&= \sum_a \big(-Q_0^{(1)}(-e^{2\pi i a}\zeta)\big)^s \quad \Big(Q_0^{(1)}(\zeta)=(1/\hat{p})\sum_{n\in\mathbb{Z}} v_n^{(1)}\zeta^{-n/\hat{p}-1}\Big) \\
&= \frac{1}{\hat{p}^{s-1}} \sum_{n\in\mathbb{Z}} \sum_{n_1\cdots+n_s=n\hat{p}} v_{n_1}^{(1)}\cdots v_{n_s}^{(1)}\,(-1)^n\,\zeta^{-n-s}.
\end{aligned} \tag{5.20}$$

Only the terms with $n\in 2\mathbb{Z}$ survive in the last expression since $v_{n_r}^{(1)}=0$ for odd n_r (due to the C$\times\tilde{\mathrm{C}}$ invariance), and $\hat{p}$ is odd now. Thus we have

$$\begin{aligned}
&= \frac{1}{\hat{p}^{s-1}} \sum_{n\in\mathbb{Z}} \sum_{n_1\cdots+n_s=n\hat{p}} v_{n_1}^{(1)}\cdots v_{n_s}^{(1)}\,\zeta^{-n-s} \\
&= \sum_a \big(Q_a^{(1)}(\zeta)\big)^s.
\end{aligned} \tag{5.21}$$

5.2 Super Kazakov series ($\hat{p}=1$)

In this case, the disk amplitude is expressed as

$$Q^{(i)}(\zeta) \equiv \Big\langle \partial\varphi^{(i)}(\zeta)\Big\rangle^{(h=0)} = \sum_{n=1}^{1+\hat{q}} n\, b_n^{(i)}\zeta^{n-1} + \sum_{n\geq 1} v_n^{(i)}\,\zeta^{-n-1}. \tag{5.22}$$

The constraints $W^1_n|\phi\rangle = 0$ $(n \geq 0)$ imply $v^{[n]}_n = 0$, and the perturbations with respect to unphysical variables $b^{[n]}_n$ can always be absorbed into a shift of $Q^{(i)}$ as in the bosonic case [9],

$$Q^{(1)}(\zeta) \to Q^{(1)}(\zeta) + \sum_{n=1}^{1+\hat{q}} n\, b^{[n]}_n \zeta^{n-1} = Q^{(1)}(\zeta) + \frac{1}{2}\, a_1(\zeta), \tag{5.23}$$

$$Q^{(2)}(\zeta) \to Q^{(2)}(\zeta) + \sum_{n=1}^{1+\hat{q}} n\, (-1)^n b^{[n]}_n \zeta^{n-1} = Q^{(2)}(\zeta) - \frac{1}{2}\, a_1(-\zeta), \tag{5.24}$$

so that one can set $a_1(\zeta) = 0$ $(\Leftrightarrow b^{[n]}_n = 0)$ without loss of generality. Then it holds that $Q^{(1)}(\zeta) = Q^{(2)}(-\zeta) \equiv Q(\zeta)$,[22] and we have

$$a_2(\zeta) = \frac{1}{2}\left[\left(Q^{(1)}(\zeta)\right)^2 + \left(-Q^{(2)}(-\zeta)\right)^2\right]_{\rm pol} = \left[\left(Q(\zeta)\right)^2\right]_{\rm pol}. \tag{5.25}$$

Thus, we have the curve

$$F(\zeta, Q) = Q^2 - a_2(\zeta) = 0. \tag{5.26}$$

1-cut solutions. Due to the general consideration given in the previous subsection, 1-cut solutions are given by the state $|\Phi\rangle$ satisfying the $W_{1+\infty}$ constraints separately (at least perturbatively),

$$(W^{(1)})^s_n|\Phi\rangle = 0, \quad (W^{(2)})^s_n|\Phi\rangle = 0 \qquad (s \geq 1;\ n \geq -s+1). \tag{5.27}$$

In particular, this implies that

$$F^{(i)}(\zeta, Q) = Q - a^{(i)}_1(\zeta) = Q - \sum_{n=1}^{1+\hat{q}} n\, b^{(i)}_n \zeta^{n-1} = 0 \quad (i = 1, 2), \tag{5.28}$$

and we have

$$Q(\zeta) = \sum_{n=1}^{1+\hat{q}} n\, b^{(i)}_n \zeta^{n-1} \quad \text{(1-cut solutions)}. \tag{5.29}$$

Note that all the one-point functions vanish, $v^{(i)}_n = 0$.

2-cut solutions. Based on the analysis of the $\hat{p} = 1$ string equations, we take boundary conditions such that 2-cut solutions have cuts in the complex ζ plane, each of which has one of its end points at infinity.

The master field equation is now given by

$$a_2(\zeta) = \left[\left(Q(\zeta)\right)^2\right]_{\rm pol} = \left(\sum_{n=1}^{\hat{q}+1} n\, b_n \zeta^{n-1}\right)^2 + f(\zeta), \tag{5.30}$$

[22] In the $\hat{p} = 1$ case with the condition $b^{[n]}_n = 0$, the two-component system can be completely written by using a single untwisted boson $\partial\phi(\lambda) = \partial\varphi(\zeta) \equiv \partial\varphi^{(1)}(\zeta) = \partial\varphi^{(2)}(-\zeta)$, as was pointed out in [20].

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where $f(\zeta)$ is a polynomial of degree $\hat{q}-2$ depending on the one-point functions $v_n^{(i)}$, and has $\hat{q}-1$ degrees of freedom. They can be adjusted by properly choosing $v_n^{(i)}$ such that the equations

$$F(\zeta^*, Q^*) = \frac{\partial F}{\partial \zeta}(\zeta^*, Q^*) = \frac{\partial F}{\partial Q}(\zeta^*, Q^*) = 0 \tag{5.31}$$

have its maximal $\hat{q}-1$ solutions. This is equivalent to the condition that $a_2(\zeta)$ has $\hat{q}-1$ solutions to the equations

$$a_2(\zeta^*) = \frac{\partial a_2}{\partial \zeta}(\zeta^*) = 0. \tag{5.32}$$

The solutions are given by[23]

$$Q(\zeta) = E(\zeta) \cdot \sqrt{\zeta^2 + a^2} \tag{5.33}$$

with a polynomial $E(\zeta) = \text{const.} \prod_{r=1}^{\hat{q}-1}(\zeta - \zeta_r^*)$.

5.3 Conformal backgrounds

1-cut solutions. The conformal backgrounds for 1-cut solutions are given by C-invariant backgrounds $b_n^{(1)} = b_n^{(2)} = b_n$ with

$$b_n = \begin{cases} -\beta \, \dfrac{\hat{p}\,\hat{q}}{n} \, \dfrac{2^{(\hat{q}-\hat{p})/\hat{p}}}{2m\hat{p} - \hat{q}} \dbinom{2m - \hat{q}/\hat{p}}{m} \left(\dfrac{a^{\hat{p}}}{4}\right)^m & \left(n = \hat{q} + \hat{p} - 2m\hat{p}\,;\; 0 \le m \le \left[\dfrac{\hat{q}+\hat{p}-1}{2\hat{p}}\right]\right) \\ 0 & \text{(otherwise)} \end{cases},$$
$$(\beta: \text{ numerical constant}). \tag{5.34}$$

They can be solved in the same manner with that in the bosonic conformal backgrounds (see eq. (3.47) of [9]); one simply needs to replace their (p,q) by $(\hat{p},\hat{q})$. For this, the curve for $Q(\zeta) = Q^{(1)}(\zeta)$ is given by $F^{(1)}(\zeta, Q) = 0$. As is shown in [9], we have the uniformization parameter z which parametrizes (ζ, Q) as[24]

$$z = a\, \cosh \tau, \tag{5.35}$$
$$\zeta = a^{\hat{p}}\, \cosh \hat{p}\tau, \tag{5.36}$$
$$Q = \beta\, a^{\hat{q}}\, \cosh \hat{q}\tau, \tag{5.37}$$

giving the curve

$$F^{(1)}(\zeta, Q) = 2^{1-\hat{p}}\, \beta^{\hat{p}}\, a^{\hat{p}\hat{q}} \left[T_{\hat{p}}\big(Q/\beta a^{\hat{q}}\big) - T_{\hat{q}}\big(\zeta/a^{\hat{p}}\big) \right] = 0, \tag{5.38}$$

[23] The general form of the irrational function would be $\sqrt{(\zeta-a)(\zeta-b)}$, but this can always be changed to $\sqrt{\zeta^2+a^2}$ by shifting ζ properly.

[24] This z corresponds to the derivative $\partial = \partial/\partial x_1^{[0]}$ in the weak coupling limit, with the relation $z = 2^{1-1/\hat{p}}\,\partial$.

where $T_n(z)$ is the first Tchebycheff polynomials of degree n, $T_n(\cosh\tau) = \cosh(n\tau)$. Taking a branch such that $\zeta/z^{\hat{p}} > 0$ for $\mathrm{Re}\, z \to \infty$, we have

$$Q_0(\zeta) = \frac{\beta}{2}\,\Big[\big(\zeta + \sqrt{\zeta^2 - a^{2\hat{p}}}\big)^{\hat{q}/\hat{p}} + \big(\zeta - \sqrt{\zeta^2 - a^{2\hat{p}}}\big)^{\hat{q}/\hat{p}}\Big]. \tag{5.39}$$

This agrees with the curve of $\eta = -1$ FZZT branes in super Liouville theory [15] if one identifies $a = \mu^{1/2\hat{p}}$ ($\mu > 0$). One can easily see that eq. (5.29) is reproduced when $\hat{p} = 1$.

Note that the total algebraic equation $F(\zeta, Q) = 0$ is expressed as

$$F(\zeta, Q) = 2^{1-2\hat{p}}\beta^{2\hat{p}}a^{2\hat{p}\hat{q}}\Big[\,T_{2\hat{p}}\big(Q/\beta a^{\hat{q}}\big) - T_{2\hat{q}}\big(\zeta/a^{\hat{p}}\big)\Big] = 0 \tag{5.40}$$

for even models, and

$$F(\zeta, Q) = 2^{2-2\hat{p}}\beta^{2\hat{p}}a^{2\hat{p}\hat{q}}\Big[\,T_{\hat{p}}\big(Q/\beta a^{\hat{q}}\big) - T_{\hat{q}}\big(\zeta/a^{\hat{p}}\big)\Big]^2 = 0 \tag{5.41}$$

for odd models. Thus, even and odd models have the same description for their curves. By requiring that the odds models are under the $\tilde{\mathrm{C}}$ orbifolding, the curves are reduced to (5.38). This explains why odd models seemed different in Liouville theory [15].

2-cut solutions. The conformal backgrounds for 2-cut solutions are given by C-invariant backgrounds $b_n^{(1)} = b_n^{(2)} = b_n$ with

$$b_n = \begin{cases} -\beta\,\dfrac{\hat{p}\,\hat{q}}{n}\,\dfrac{2^{(\hat{q}-\hat{p})/\hat{p}}}{2m\hat{p}-\hat{q}}\dbinom{2m-\hat{q}/\hat{p}}{m}\Big(-\dfrac{a^{2\hat{p}}}{4}\Big)^m & \Big(n = \hat{q}+\hat{p}-2m\hat{p}\,;\ 0 \le m \le \Big[\dfrac{\hat{q}+\hat{p}-1}{2\hat{p}}\Big]\Big) \\ 0 & (\text{otherwise}) \end{cases}$$
$$(\beta:\ \text{numerical constant}). \tag{5.42}$$

Note that they could lead to C- and $\tilde{\mathrm{C}}$-invariant backgrounds if we took $\hat{p} + \hat{q} \in 2\mathbb{Z}$. Thus, conformal backgrounds are allowed to have 2-cut solutions only for even minimal superstrings, $(p, q) = (2\hat{p}, 2\hat{q})$ with $\hat{p} + \hat{q} \in 2\mathbb{Z} + 1$, which we assume in the following discussions.

For this, the curve $F(\zeta, Q) = 0$ for $Q(\zeta) = Q^{(1)}(\zeta)$ is not factorized, and is parametrized as

$$z = a\,\cosh\tau, \tag{5.43}$$
$$\zeta = a^{\hat{p}}\,\sinh\hat{p}\tau, \tag{5.44}$$
$$Q = \beta\,a^{\hat{q}}\,\sinh\hat{q}\tau, \tag{5.45}$$

giving a curve

$$F(\zeta, Q) = (-1)^{\hat{p}}\,2^{1-2\hat{p}}\,\beta^{2\hat{p}}\,a^{2\hat{p}\hat{q}}\Big[\,T_{2\hat{p}}(iQ/\beta a^{\hat{q}}) + T_{2\hat{q}}(i\zeta/a^{\hat{p}})\Big] = 0 \quad (\hat{p} + \hat{q} \in 2\mathbb{Z} + 1). \tag{5.46}$$

Taking a branch such that $\zeta/z^{\hat{p}} > 0$ for $\mathrm{Re}\, z \to \infty$, we have

$$Q_0(\zeta) = \frac{\beta}{2}\,\Big[\big(\zeta + \sqrt{\zeta^2 + a^{2\hat{p}}}\big)^{\hat{q}/\hat{p}} - \big(-\zeta + \sqrt{\zeta^2 + a^{2\hat{p}}}\big)^{\hat{q}/\hat{p}}\Big]. \tag{5.47}$$

This agrees with the curve of $\eta = -1$ FZZT branes in super Liouville theory [15] if one identifies $a = |\mu|^{1/2\hat{p}}$ ($\mu < 0$). Note that eq. (5.33) is reproduced when $\hat{p} = 1$.

6. $\eta = -1$ ZZ branes

In order to make calculations definite, we take the conformal backgrounds (5.34) for 1-cut solutions and (5.42) for 2-cut solutions. Then the saddle points in (4.22) describe ZZ branes [8, 9]. Recalling that the relations $Q_a^{(1)}(\zeta) = Q_a^{(2)}(\zeta)$ hold in the conformal backgrounds, we have

$$\text{neutral D-instantons:}\quad \Gamma_{ab}^{(11)}(\zeta) = \Gamma_{ab}^{(22)}(-\zeta) = \big\langle \varphi_a^{(1)}(\zeta)\big\rangle^{(h=0)} - \big\langle \varphi_b^{(1)}(\zeta)\big\rangle^{(h=0)} \quad (a \neq b), \tag{6.1}$$

$$\text{charged D-instantons:}\quad \Gamma_{ab}^{(12)}(\zeta) = \Gamma_{ab}^{(21)}(-\zeta) = \big\langle \varphi_a^{(1)}(\zeta)\big\rangle^{(h=0)} - \big\langle \varphi_b^{(1)}(-\zeta)\big\rangle^{(h=0)} \quad (\forall a, \forall b). \tag{6.2}$$

6.1 Neutral ZZ branes

The $\Gamma_{ab}^{(11)}$ is expressed as a function of τ by integrating $d\Gamma_{ab}^{(11)}/d\tau = (d\Gamma_{ab}^{(11)}/d\zeta)\cdot(d\zeta/d\tau)$. Noting that

$$\frac{d\Gamma_{ab}^{(11)}}{d\zeta} = Q_a^{(1)}\big(\zeta(\tau)\big) - Q_b^{(1)}\big(\zeta(\tau)\big) = Q_0^{(1)}\big(\zeta(\tau_a)\big) - Q_0^{(1)}\big(\zeta(\tau_b)\big) \tag{6.3}$$

with $\tau_a \equiv \tau + 2\pi i a/\hat{p}$, we obtain

$$\Gamma_{ab}^{(11)} = \beta\,\hat{p}\,a^{\hat{p}+\hat{q}}\Big[\frac{1}{\hat{q}+\hat{p}}\Big(\cosh(\hat{p}\tau+\hat{q}\tau_a) - \cosh\big(\hat{p}\tau+\hat{q}\tau_b\big)\Big)\mp \mp\frac{1}{\hat{q}-\hat{p}}\Big(\cosh(\hat{p}\tau-\hat{q}\tau_a) - \cosh\big(\hat{p}\tau-\hat{q}\tau_b\big)\Big)\Big], \tag{6.4}$$

where the upper (*resp.* lower) sign corresponds to 1-cut (*resp.* 2-cut) solutions. The saddle points are found by solving (6.3) as in [8, 9]:[25]

$$\tau^* = \begin{cases} \dfrac{1}{2}\Big(-\dfrac{2(a+b)}{\hat{p}} + \dfrac{2k}{\hat{q}}\Big)\pi i & \text{(1-cut)} \\ \dfrac{1}{2}\Big(-\dfrac{2(a+b)}{\hat{p}} + \dfrac{2k+1}{\hat{q}}\Big)\pi i & \text{(2-cut)} \end{cases} \quad (k \in \mathbb{Z}). \tag{6.5}$$

Setting $m = 2(b-a)$ for both of 1-cut and 2-cut solutions, and $n = 2k$ for 1-cut and $n = 2k+1$ for 2-cut solutions, we have

$$\tau_a^* = \frac{1}{2}\Big(-\frac{m}{\hat{p}} + \frac{n}{\hat{q}}\Big)\pi i, \tag{6.6}$$

$$\tau_b^* = \frac{1}{2}\Big(+\frac{m}{\hat{p}} + \frac{n}{\hat{q}}\Big)\pi i. \tag{6.7}$$

Taking into account the fact that distinct ZZ branes are labeled by the indices (m, n) with $\hat{q}m - \hat{p}n \geq 0$ and also counting the contributions from $\Gamma_{ab}^{(22)}$, the number of neutral ZZ

[25] Other saddle points corresponding to $d\zeta/d\tau = 0$ do not give major contributions to the contour integral, since the Jacobian vanishes there [8].

branes is given by

$$\begin{cases} \dfrac{(\hat{p}-1)(\hat{q}-1)}{2} \times 2 & \text{(1-cut)} \\ \Big[\dfrac{(\hat{p}-1)\hat{q}+1}{2}\Big] \times 2 & \text{(2-cut)}, \end{cases} \tag{6.8}$$

where $[\cdots]$ denotes the integer part.[26] Substituting the values (6.5)–(6.7) into Γ_{ab}, we obtain the partition functions of neutral ZZ branes:

$$\Gamma^{(11)}_{ab}(\tau^*) = -(-1)^{m+n}\,\frac{2\,\beta\,\hat{q}\,\hat{p}}{\hat{q}^2-\hat{p}^2}\,a^{\hat{q}+\hat{p}}\,\sin\frac{m(\hat{q}-\hat{p})}{2\hat{p}}\pi\cdot\sin\frac{n(\hat{q}-\hat{p})}{2\hat{q}}\pi. \tag{6.9}$$

They completely agree with the analysis of [15].

6.2 Charged ZZ branes

The $\Gamma^{(12)}_{ab}$ is expressed as a function of τ by integrating $d\Gamma^{(12)}_{ab}/d\tau = (d\Gamma^{(12)}_{ab}/d\zeta)\cdot(d\zeta/d\tau)$. Noting that

$$\frac{d\Gamma^{(12)}_{ab}}{d\zeta} = Q^{(1)}_a\big(\zeta(\tau)\big) + Q^{(1)}_b\big(-\zeta(\tau)\big) = Q^{(1)}_0\big(\zeta(\tau_a)\big) + Q^{(1)}_0\big(\zeta(\bar{\tau}_b)\big) \tag{6.10}$$

with $\tau_a \equiv \tau + 2\pi i a/\hat{p}$ and $\bar{\tau}_b \equiv \tau + \pi i\,(2b+1)/\hat{p}$, we obtain

$$\begin{aligned} \Gamma^{(12)}_{ab} = \beta\,\hat{p}\,a^{\hat{p}+\hat{q}}\Big[&\frac{1}{\hat{q}+\hat{p}}\Big(\cosh(\hat{p}\tau+\hat{q}\tau_a)+\cosh\big(\hat{p}\tau+\hat{q}\bar{\tau}_b\big)\Big)\mp \\ &\mp\frac{1}{\hat{q}-\hat{p}}\Big(\cosh(\hat{p}\tau-\hat{q}\tau_a)+\cosh\big(\hat{p}\tau-\hat{q}\bar{\tau}_b\big)\Big)\Big], \end{aligned} \tag{6.11}$$

where the upper (*resp.* lower) sign corresponds to 1-cut (*resp.* 2-cut) solutions. The saddle points are found similarly by solving (6.10) as

$$\tau^* = \begin{cases} \dfrac{1}{2}\Big(-\dfrac{2(a+b)-1}{\hat{p}}+\dfrac{2k+1}{\hat{q}}\Big)\pi i & \text{(1-cut)} \\ \dfrac{1}{2}\Big(-\dfrac{2(a+b)-1}{\hat{p}}+\dfrac{2k}{\hat{q}}\Big)\pi i & \text{(2-cut)} \end{cases} \qquad (k\in\mathbb{Z}). \tag{6.12}$$

Setting $m = 2(b-a)+1$ for both of 1-cut and 2-cut solutions, and $n = 2k+1$ for 1-cut and $n = 2k$ for 2-cut solutions, we have

$$\tau^*_a = \frac{1}{2}\Big(-\frac{m}{\hat{p}}+\frac{n}{\hat{q}}\Big)\pi i, \tag{6.13}$$

$$\bar{\tau}^*_b = \frac{1}{2}\Big(+\frac{m}{\hat{p}}+\frac{n}{\hat{q}}\Big)\pi i. \tag{6.14}$$

The number of these charged ZZ branes can be counted in a similar way, and we obtain

$$\begin{cases} \Big[\dfrac{\hat{p}\hat{q}+1}{2}\Big] \times 2 & \text{(1-cut)} \\ \Big[\dfrac{\hat{p}(\hat{q}-1)+1}{2}\Big] \times 2 & \text{(2-cut)}. \end{cases} \tag{6.15}$$

[26]We also count an instanton with $\hat{q}m-\hat{p}n=0$, i.e. $(m,n)=(\hat{p},\hat{q})$.

Substituting the values (6.12)–(6.14) into Γ_{ab}, we obtain the partition functions of charged ZZ branes:

$$\Gamma^{(12)}_{ab}(\tau^*) = -(-1)^{m+n}\,\frac{2\,\beta\,\hat{q}\,\hat{p}}{\hat{q}^2-\hat{p}^2}\,a^{\hat{q}+\hat{p}}\,\sin\frac{m(\hat{q}-\hat{p})}{2\hat{p}}\pi\cdot\sin\frac{n(\hat{q}-\hat{p})}{2\hat{q}}\pi. \qquad (6.16)$$

They also agree with the analysis of [15].

7. Conclusion and discussion

In this paper, we formulate a string field theory of type 0B (p,q) minimal superstrings, based on the Douglas equation of two-cut two-matrix models. We find that the 2cKP hierarchy is the underlying integrable structure in these systems, and solutions are given by decomposable fermion states which satisfy the $W_{1+\infty}$ constraints. We show that the 2cKP hierarchy is reducible for backgrounds invariant under the $\mathbb{Z}_2$ transformation, $\tilde{\mathrm{C}} : (\boldsymbol{\sigma},\boldsymbol{L}) \to (-\boldsymbol{\sigma},-\boldsymbol{L})$, which is the case for odd minimal superstrings. Taking into account this reduction, we establish the correspondence between the operators in the 2cKP hierarchy and those of super Liouville theory. We also show that background R-R fluxes are naturally incorporated in our treatment without ambiguity. We calculate various correlation functions and show that they all agree with the results of super Liouville theory.

Type 0A superstrings are obtained by orbifolding the above type 0B systems with another $\mathbb{Z}_2$ symmetry $C : (\boldsymbol{\sigma},\boldsymbol{L}) \to (-\boldsymbol{\sigma},\boldsymbol{L})$ (charge conjugation). This can be carried out straightforwardly in the absence of R-R flux. However, in the presence of R-R fluxes, we better move to the description using Toda lattice hierarchy. The investigation in this direction is in progress, and will be reported in our future communication [54].

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A. Derivation of eq. (2.13)

To study the double scaling limit, we interpret the operators $\boldsymbol{Q_1}$, $\boldsymbol{Q_2},\dots$ as the following difference operators with respect to the index $n=0,1,\cdots$:

$$\langle\alpha_n|\boldsymbol{Q_1}^{\mathrm{T}} = \sum_m (\boldsymbol{Q_1}^{\mathrm{T}})_{nm}\langle\alpha_m| = \sum_{k=m-n}(\boldsymbol{Q_1}^{\mathrm{T}})_{n,n+k}\,e^{k\partial_n}\langle\alpha_n| \equiv \sum_k a_k(n)\,\Lambda^k\langle\alpha_n|, \quad (A.1)$$

where $a_k(n) \equiv (\boldsymbol{A}^{\mathrm{T}})_{n,n+k}$ and $\Lambda\,f(n) = f(n+1)$. Equation (2.11) or (2.12)) then gives the recursion relations for the sequence $\big\{a_k(n)\big\}_{n=0}^{\infty}$ $(k\in\mathbb{Z})$.

In the vicinity of the point $(x_c, y_c) = (0, 0)$ we consider, the orthonormal polynomials $\{\alpha_n(x),\, \beta_n(y)\}_{n=0,1,2,\cdots}$ do not behave smoothly for a small change of n, and thus we need to redefine them as follows [44]:

$$\begin{pmatrix} \widetilde{\alpha}^{(e)}_{2k}(x) \\ \widetilde{\alpha}^{(o)}_{2k+1}(x) \end{pmatrix} \equiv (-1)^k \begin{pmatrix} \alpha_{2k}(x) \\ \alpha_{2k+1}(x) \end{pmatrix}, \quad \begin{pmatrix} \widetilde{\beta}^{(e)}_{2k}(x) \\ \widetilde{\beta}^{(o)}_{2k+1}(x) \end{pmatrix} \equiv (-1)^k \begin{pmatrix} \beta_{2k}(x) \\ \beta_{2k+1}(x) \end{pmatrix}. \tag{A.2}$$

With this new basis, a smooth operator acting on orthonormal polynomials can be expressed as a 2×2 matrix of the following form:

$$\widetilde{\boldsymbol{Q}_1} \equiv (-1)^{[\frac{\boldsymbol{n}}{2}]} \boldsymbol{Q}_1 (-1)^{[\frac{\boldsymbol{n}}{2}]} = \sum_{l=0}^{\infty} (-1)^l \begin{pmatrix} a_{-2l}(2k)\,\Lambda^{-2l} & a_{-2l+1}(2k+1)\,\Lambda^{-2l+1} \\ -a_{-2l+1}(2k)\,\Lambda^{-2l+1} & a_{-2l}(2k+1)\,\Lambda^{-2l} \end{pmatrix}, \tag{A.3}$$

where $\boldsymbol{n} = (n\,\delta_{mn})$, and the index is rearranged to run as $(0, 2, 4, \cdots; 1, 3, 5, \cdots)$.

For a generic case, the even and odd subsequences ($\{a_l(2k)\}_{k=0}^{\infty}$ and $\{a_l(2k+1)\}_{k=0}^{\infty}$) are allowed to have different large N limits. At the 2-cut critical points, however, since the system carries a continuum phase transition to 1-cut phase, we assume that both of the even/odd subsequences have the same limit a^c_l. Thus the leading large N behaviour of $\widetilde{\boldsymbol{Q}_1}$ can be given as

$$\widetilde{\boldsymbol{Q}_1} \sim \sum_{l=0}^{\infty} (-1)^l \Big(a^c_{-2l}\,\Lambda^{-2l} - i\sigma_2\, a^c_{-2l+1}\,\Lambda^{-2l+1} \Big). \tag{A.4}$$

If we assume that this behaves as $\sim (\Lambda - 1)^{\hat{p}}$ with some integer $\hat{p} \in \mathbb{N}$, $\widetilde{\boldsymbol{Q}_1}$ becomes a differential operator of order $\hat{p}$. The same also holds for $\widetilde{\boldsymbol{Q}_2}$, which we assume to give a differential operator of order $\hat{q}$. The correctness of this assumption can be checked by using eq. (2.12) with the corresponding critical potentials $V_1(x)$ and $V_2(y)$, and can actually be shown by solving the linear equations for the coefficients of potentials as in the case of bosonic strings [31].

B. Charge conjugation in terms of infinite dimensional Grassmannian

The charge conjugation is realized by exchanging the operators $\psi^{(1)}(\lambda)$ and $\psi^{(2)}(\lambda)$. This transformation is well described in terms of the Sato-Wilson operator $\boldsymbol{W}(x;\partial)$. It is defined by the similarity transformation with σ_1 as

$$\boldsymbol{W}(x;\partial) \to \boldsymbol{W}^{\mathrm{C}}(x^{\mathrm{C}};\partial) \equiv \sigma_1 \boldsymbol{W}(x;\partial)\sigma_1. \tag{B.1}$$

Under this transformation, the Sato equations are invariant,

$$g\,\frac{\partial \boldsymbol{W}^{\mathrm{C}}(x^{\mathrm{C}};\partial)}{\partial x^{\mathrm{C}}_n} = -\big(\boldsymbol{W}^{\mathrm{C}} \sigma_3^{\mu} \partial^n \boldsymbol{W}^{\mathrm{C}-1}\big)_{-} \boldsymbol{W}^{\mathrm{C}}(x^{\mathrm{C}};\partial). \tag{B.2}$$

The pair of differential operators $(\boldsymbol{P}, \boldsymbol{Q})$ at the background b transforms as

$$\big(\boldsymbol{P}(b), \boldsymbol{Q}(b)\big) \to \big(-\boldsymbol{P}^{\mathrm{C}}(b^{\mathrm{C}}), -\boldsymbol{Q}^{\mathrm{C}}(b^{\mathrm{C}})\big), \tag{B.3}$$

and eq. (3.8) is changed into the following:

$$\boldsymbol{P}^{\rm C}\,\tilde\Psi^{\rm C}(x^{\rm C};\zeta)=\tilde\Psi^{\rm C}(x^{\rm C};\zeta)\begin{pmatrix}-\zeta & 0\\ 0 & \zeta\end{pmatrix},\quad \boldsymbol{Q}^{\rm C}\,\tilde\Psi^{\rm C}(x^{\rm C};\zeta)=g\,\tilde\Psi^{\rm C}(x^{\rm C};\zeta)\begin{pmatrix}-\overleftarrow{\frac{\partial}{\partial\zeta}} & 0\\ 0 & \overleftarrow{\frac{\partial}{\partial\zeta}}\end{pmatrix} \tag{B.4}$$

with $\Psi^{\rm C}(x^{\rm C};\zeta)\equiv\sigma_1\Psi(x^{\rm C};\zeta)\sigma_1$.

We now turn to the free-fermion description. The charge conjugation of a decomposable fermion state $\big|\Phi\big\rangle$ is given by[27]

$$\begin{aligned}\big|\Phi\big\rangle&=\prod_{k=0}^{\infty}\prod_{i=1}^{2}\Big[\oint\frac{d\lambda}{2\pi i}\,\boldsymbol{\Phi}_k^{(i)}(x=0;\lambda)\,\bar\psi(\lambda)\Big]\big|\infty\big\rangle\to\prod_{k=0}^{\infty}\prod_{i=1}^{2}\Big[\oint\frac{d\lambda}{2\pi i}\,\boldsymbol{\Phi}_k^{{\rm C}(i)}(x^{\rm C}=0;\lambda)\,\bar\psi(\lambda)\Big]\big|\infty\big\rangle\\&=\prod_{k=0}^{\infty}\prod_{i=1}^{2}\Big[\oint\frac{d\lambda}{2\pi i}\,\boldsymbol{\Phi}_k^{(i)}(x=0;\lambda)\,\sigma_1\bar\psi(\lambda)\Big]\big|\infty\big\rangle\equiv\big|\Phi^{\rm C}\big\rangle,\end{aligned} \tag{B.5}$$

and we see that the two fermions, $\psi^{(1)}(\lambda)$ and $\psi^{(2)}(\lambda)$, are actually exchanged.

The charge conjugation is not the symmetry of the theory and changes the background of the theory in general. If we consider, however, the $\mathbb{Z}_2$ symmetric matrix models, $w(-x,-y)=w(x,y)$, this system is $\mathbb{Z}_2$ symmetric at least when their two Fermi levels are equal to each other. So we can require the following $\mathbb{Z}_2$ symmetric property:

$$\big|\Phi^{\rm C}\big\rangle=\big|\Phi\big\rangle \tag{B.6}$$

at least perturbatively.

C. Derivation of string equations

In this appendix we derive the string equations (3.40)–(3.42) from the Virasoro constraints (3.39). First we introduce a linear differential operator $\mathcal{O}$ as

$$\mathcal{O}\equiv\sum_{n\geq1}(n+\hat p)\big(x_{n+\hat p}^{(1)}\partial_n^{(1)}-x_{n+\hat p}^{(2)}\partial_n^{(2)}\big), \tag{C.1}$$

then (3.39) can be rewritten as

$$\mathcal{O}\ln\tau_\nu(x)=-\frac{1}{2g^2}\sum_{n=1}^{\hat p-1}n(\hat p-n)\big(x_{\hat p-n}^{(1)}x_n^{(1)}-x_{\hat p-n}^{(2)}x_n^{(2)}\big)-\frac{\hat p\nu}{g}\big(x_{\hat p}^{(1)}+x_{\hat p}^{(2)}\big), \tag{C.2}$$

which takes for $\hat p=1$, 2 as

$$\hat p=1\ :\quad \mathcal{O}\ln\tau_\nu(x)=-\frac{\nu}{g}\,\big(x_1^{(1)}+x_1^{(2)}\big), \tag{C.3}$$

$$\hat p=2\ :\quad \mathcal{O}\ln\tau_\nu(x)=-\frac{1}{2g^2}\Big((x_1^{(1)})^2-(x_1^{(2)})^2\Big)-\frac{2\nu}{g}\,\big(x_2^{(1)}+x_2^{(2)}\big). \tag{C.4}$$

[27]Recall that $\bar\psi(\lambda)$ is a column vector, $\bar\psi(\lambda)=\big(\bar\psi^{(1)}(\lambda),\bar\psi^{(2)}(\lambda)\big)^{\rm T}$.

Furthermore, taking a difference between eq. (C.1) with $\nu \pm 1$ and that with ν, we obtain the following formula for $H_\pm(x) = \tau_{\nu\pm1}(x)/\tau_\nu(x)$:

$$\mathcal{O}\,\ln H_\pm(x) = \mp\frac{\hat{p}}{g}\,(x^{(1)}_{\hat{p}} + x^{(2)}_{\hat{p}}). \tag{C.5}$$

Then we obtain string equations as follows. First for the $\hat{p}=1$ case, by using eqs. (2.61), (2.62) and (2.64) we obtain

$$\begin{aligned}
&\sum_{n\geq1}(n+1)\Big(x^{(1)}_{n+1}(\boldsymbol{e}^{(1)}\boldsymbol{L}^n)_{-1} - x^{(2)}_{n+1}(\boldsymbol{e}^{(2)}\boldsymbol{L}^n)_{-1}\Big) = -g\,\mathcal{O}w_1\\
&= -g\,\mathcal{O}\cdot\begin{pmatrix} -g\,\partial^{(1)}_1\ln\tau_\nu & -H_+\\ H_- & -g\,\partial^{(2)}_1\ln\tau_\nu\end{pmatrix} = \begin{pmatrix} g^2\,\partial^{(1)}_1\mathcal{O}\ln\tau_\nu & g\,H_+\,\mathcal{O}\ln H_+\\ -g\,H_-\,\mathcal{O}\ln H_- & g^2\,\partial^{(2)}_1\mathcal{O}\ln\tau_\nu\end{pmatrix}\\
&= \begin{pmatrix} -g\,\nu & -H_+(x^{(1)}_1 + x^{(2)}_1)\\ -H_-(x^{(1)}_1 + x^{(2)}_1) & -g\,\nu\end{pmatrix} = -g\,\nu\boldsymbol{1}_2 - \Big(x^{(1)}_1(\boldsymbol{e}^{(1)})_{-1} - x^{(2)}_1(\boldsymbol{e}^{(2)})_{-1}\Big),
\end{aligned} \tag{C.6}$$

from which we obtain the string equation (3.40). In deriving the last expression, we used the formula (3.43).

As for the $\hat{p}=2$ case, by using eq. (2.61) and (2.64) again we have

$$\begin{aligned}
\sum_{n\geq1}(n+2)\big(x^{(1)}_{n+2}(\boldsymbol{e}^{(1)}\boldsymbol{L}^n)_{-1} - x^{(2)}_{n+2}(\boldsymbol{e}^{(2)}\boldsymbol{L}^n)_{-1}\big) &= \begin{pmatrix} -x^{(1)}_1 & -2H_+(x^{(1)}_1 + x^{(2)}_1)\\ -2H_-(x^{(1)}_1 + x^{(2)}_1) & x^{(2)}_1\end{pmatrix}\\
&= -2\Big(x^{(1)}_2(\boldsymbol{e}^{(1)})_{-1} - x^{(2)}_2(\boldsymbol{e}^{(2)})_{-1}\Big) - \Big(x^{(1)}_1(\boldsymbol{e}^{(1)}\boldsymbol{L}^{-1})_{-1} - x^{(2)}_1(\boldsymbol{e}^{(2)}\boldsymbol{L}^{-1})_{-1}\Big),
\end{aligned} \tag{C.7}$$

from which we obtain the first string equation (3.41). The second string equation (3.42) is obtained with a little algebra, but in a way similar to the above:

$$\begin{aligned}
&\sum_{n\geq1}(n+2)\big(x^{(1)}_{n+2}(\boldsymbol{e}^{(1)}\boldsymbol{L}^n)_{-2} - x^{(2)}_{n+2}(\boldsymbol{e}^{(2)}\boldsymbol{L}^n)_{-2}\big) = -g\,\mathcal{O}w_1 + g\,\mathcal{O}w_1\cdot w_1\\
&= -g\begin{pmatrix}\nu-1/2 & 0\\ 0 & \nu+1/2\end{pmatrix} + 2(x^{(1)}_2 + x^{(2)}_2)\begin{pmatrix} H_+H_- & g\,\partial^{(2)}_1 H_+\\ g\,\partial^{(1)}_1 H_- & -H_+H_-\end{pmatrix} - (x^{(1)}_1 + x^{(2)}_1)\begin{pmatrix} 0 & H_+\\ H_- & 0\end{pmatrix}\\
&= -g\begin{pmatrix}\nu-1/2 & 0\\ 0 & \nu+1/2\end{pmatrix} - 2\Big[x^{(1)}_2(\boldsymbol{e}^{(1)})_{-2} - x^{(2)}_2(\boldsymbol{e}^{(2)})_{-2}\Big]\\
&\qquad\qquad - \Big[x^{(1)}_1(\boldsymbol{e}^{(1)}\boldsymbol{L}^{-1})_{-2} - x^{(2)}_1(\boldsymbol{e}^{(2)}\boldsymbol{L}^{-1})_{-2}\Big],
\end{aligned} \tag{C.8}$$

where we have used the formulas (3.44) and (3.46).

D. ZS hierarchy and $\hat{p}=1$ superstrings

Critical behaviour of 2-cut one matrix models have been studied extensively, and a series of string equations are obtained explicitly [17–25]. We here rederive the results from the viewpoint of 2cKP hierarchy, and compare them with our analysis based on the string field formulation.

We first note that any 2-cut one-matrix model can be realized as a 2-cut two-matrix model by taking $V_2(y) = cy^2/2$ in the potential $w(x,y) = V_1(x) + V_2(y) - cxy$. Then the differential operator $\boldsymbol{P}$ is of order one and will have the form

$$\boldsymbol{P} = \sigma_3\,\partial + 2H, \qquad H \equiv \begin{pmatrix} 0 & H_+ \\ H_- & 0 \end{pmatrix}, \tag{D.1}$$

where $H_\pm \in i\,\mathbb{R}$ due to the reality condition (2.15). This corresponds to the so-called Zakharov-Shabat hierarchy [20, 21] and was proposed to describe $\hat{p} = 1$ minimal superstrings in [24]. The reduction condition $\boldsymbol{P} = \boldsymbol{\sigma L} = (\boldsymbol{\sigma L})_+$ is in our language the $\hat{p} = 1^{\text{st}}$ reduction of 2cKP hierarchy with $H_\pm = \tau_{\nu\pm1}/\tau_\nu$. The pseudo-differential operator $\boldsymbol{\sigma} = \sigma_3 + 2\sum_{n=1}^{\infty} H_n\,\partial^{-n}$ can be easily obtained by solving the conditions $[\boldsymbol{P}, \boldsymbol{\sigma}] = 0$ and $\boldsymbol{\sigma}^2 = \mathbf{1}_2$ as

$$\begin{aligned}
H_1 &= H, \\
H_2 &= -\frac{1}{2}\partial H - \sigma_3 H^2, \\
H_3 &= \frac{3}{2}\sigma_3 H\partial H + \frac{1}{2}\sigma_3 \partial H\cdot H + \frac{1}{4}\partial^2 H - 2H^3, \\
H_4 &= -\frac{7}{4}\sigma_3 H\partial^2 H - \frac{5}{4}\sigma_3(\partial H)^2 - \frac{1}{4}\sigma_3\partial^2 H\cdot H + 3\sigma_3 H^4 + \\
&\qquad + 6H^2\partial H + 3H\partial H\cdot H - \frac{1}{8}\partial^3 H, \ \cdots .
\end{aligned} \tag{D.2}$$

From this one finds

$$(\boldsymbol{\sigma})_{-1} = 2H, \tag{D.3}$$

$$(\boldsymbol{\sigma P})_{-1} = 2H^2 - \partial H, \tag{D.4}$$

$$(\boldsymbol{\sigma P}^2)_{-1} = \sigma_3\cdot(\partial H\cdot H - H\cdot\partial H) + \frac{1}{2}\,\partial^2 H + 4H^3, \tag{D.5}$$

$$(\boldsymbol{\sigma P}^3)_{-1} = \frac{1}{2}\,\partial^2 H\cdot H + 6H^4 + \frac{1}{2}\,H\cdot\partial^2 H - \frac{1}{2}\,(\partial H)^2 + \sigma_3\left(\frac{1}{4}\,\partial^3 H + 6H^2\cdot\partial H\right). \tag{D.6}$$

The differential operator $\boldsymbol{Q}$ is then given by

$$\boldsymbol{Q} = \sum_{n=1}^{\hat{q}+1} n\, b_n^{[n-1]}\,(\boldsymbol{\sigma P}^{n-1})_+ = b_1^{[0]}\sigma_3 + \sum_{n=2}^{\hat{q}+1} n\, b_n^{[n-1]}\,(\boldsymbol{\sigma P}^{n-1})_+, \tag{D.7}$$

and the corresponding string equation is obtained by rewriting the Douglas equation as[28]

$$\begin{aligned}
0 &= \sum_{n=1}^{\hat{q}+1} n\, b_n^{[n-1]}\big[\boldsymbol{P}, (\boldsymbol{\sigma P}^{n-1})_+\big] = -\sum_{n=1}^{\hat{q}+1} n\, b_n^{[n-1]}\big[\boldsymbol{P}, (\boldsymbol{\sigma P}^{n-1})_-\big] \\
&= -\sum_{n=1}^{\hat{q}+1} n\, b_n^{[n-1]}\big[\sigma_3\partial + 2H, (\boldsymbol{\sigma P}^{n-1})_{-1}\partial^{-1} + (\boldsymbol{\sigma P}^{n-1})_{-2}\partial^{-2} + \cdots\big] \\
&= -\sum_{n=1}^{\hat{q}+1} n\, b_n^{[n-1]}\big[\sigma_3, (\boldsymbol{\sigma P}^{n-1})_{-1}\big].
\end{aligned} \tag{D.8}$$

[28] Setting $b_1^{[0]} = \xi$, we have $[\sigma_3\partial, \sigma_3 b_1^{[0]}] = g\mathbf{1}_2$, and thus the Douglas equation $[\boldsymbol{P}, \boldsymbol{Q}] = g\mathbf{1}_2$ reduces to (D.8)

Since a nontrivial physical operator appears only once for each n, we use the parametrization $t_n \equiv b_n^{[n-1]}$ as in the main text.

A string equation of the $\hat{q}=2$ case is given by

$$\begin{cases} 0 = -2t_1 H_+ - 3t_3\big(\frac{1}{2}\partial^2 H_+ + 4H_+^2 H_-\big), \\ 0 = -2t_1 H_- - 3t_3\big(\frac{1}{2}\partial^2 H_- + 4H_-^2 H_+\big). \end{cases} \tag{D.9}$$

By introducing $H_\pm = \frac{i}{4}(f_1 \pm f_2)$ and taking $t_3 \equiv 8/3$ ($\beta = 4$) and $t_1 \equiv -\mu$, eqs. (D.9) are rewritten as

$$\begin{cases} 0 = \frac{1}{2}\mu f_1 - g^2 f_1'' + \frac{1}{2}f_1(f_1^2 - f_2^2), \\ 0 = \frac{1}{2}\mu f_2 - g^2 f_2'' + \frac{1}{2}f_2(f_1^2 - f_2^2). \end{cases} \tag{D.10}$$

Here $'$ implies the derivative with respect to μ, and we have used $\partial = g\,\partial/\partial t_1 = -g\,\partial/\partial\mu$. It is also convenient to rewrite them with $H_\pm \equiv \frac{i}{4} r e^{\pm\chi}$ as

$$\begin{cases} 0 = \dfrac{1}{2}\mu r - g^2 r'' - g^2 r(\chi')^2 + \dfrac{1}{2}r^3, \\ 0 = (r^2\chi')'. \end{cases} \tag{D.11}$$

Noticing that $r^2\chi'$ can be written as $-(1/g)\,(H_+\partial H_- - \partial H_+ H_-)$, one can see that the first equation of (3.51) gives an integration constant for the second equation of (D.11) as $r^2\chi' = \nu$. Taking the $\mathbb{Z}_2$ symmetric reduction ($\chi = 0$ and thus $\nu = 0$) gives the celebrated Painlevé II equation [18]:

$$0 = \frac{1}{2}\mu r - g^2 r'' + \frac{1}{2}r^3. \tag{D.12}$$

The $\hat{q}=3$ critical point exists in a flow generated by an R-R operator [24]. The string equation is given by

$$\begin{cases} 0 = -2t_1 H_+ - 4t_4\big(\frac{1}{4}\partial^3 H_+ + 6H_+ H_- \partial H_+\big), \\ 0 = -2t_1 H_- + 4t_4\big(\frac{1}{4}\partial^3 H_- + 6H_+ H_- \partial H_-\big), \end{cases} \tag{D.13}$$

and is written with $H_\pm = \frac{i}{4}(f_1 \pm f_2)$, $t_4 \equiv 4$ and $t_1 \equiv -\mu$ as

$$\begin{cases} 0 = \frac{1}{2}\mu f_1 - f_2''' + \frac{3}{2}f_2'(f_1^2 - f_2^2), \\ 0 = \frac{1}{2}\mu f_2 - f_1''' + \frac{3}{2}f_1'(f_1^2 - f_2^2). \end{cases} \tag{D.14}$$

This is the string equation found in [20, 21]. We can also rewrite it as

$$\begin{cases} 0 = \frac{1}{2}\mu r - 3(r'\chi')' - r(\chi''' + (\chi')^3) + \frac{3}{2}r^3\chi', \\ 0 = \big(rr'' - \frac{1}{2}(r')^2 + \frac{3}{2}r^2(\chi')^2 - \frac{3}{13}r^4\big)'. \end{cases} \tag{D.15}$$

The R-R flux ν again gives its integration constant.

We end this appendix with a comment on the normalization of the free energy $F(\mu;g)$. By using the formula (2.65) for $H_\pm$ together with the reduction condition $\partial_1^{[1]}\tau_\nu = 0$, we obtain

$$g^2(\partial_1^{[0]})^2 \ln\tau_\nu = g^2\frac{\partial^2}{\partial\mu^2}\ln\tau_\nu = 4H_+H_- = -\frac{1}{4}r^2. \tag{D.16}$$

It is consistent with the normalization of one-matrix-model calculation [18 − 21],

$$u(\mu; g) \equiv \frac{\partial^2}{\partial \mu^2} F(\mu; g) = \frac{1}{4} r^2(\mu; g), \tag{D.17}$$

with the identification $\mathcal{Z} = e^{-g^{-2}F(\mu;g)} = \tau_\nu(\mu; g)$ [21].

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